1. [25 pts] Consider the initial value problem

\[ -u'' = \alpha \sin u, \quad t \in [0,1] \]

where \( \alpha \) is a real parameter and the initial conditions are \( u'(0) = 1 \) and \( u(0) = 0 \). Transform this problem into a Volterra integral equation. For what values of \( \alpha \) can you prove that the solution \( u = 0 \) is unique?

OK. First of all, \( u = 0 \) is not a solution to this problem. That was my goof, and no one is held responsible for any work that was led astray by this suggestion.

The first step is to write this ODE as a Volterra integral equation. If we integrate twice and apply the initial data, we obtain

\[ u(x) = x - \alpha \int_0^x \int_s^x \sin[u(s)] \, dt \, ds. \]

It’s not a Volterra equation yet, but we can reverse the order of integration to convert it into one.

\[ u(x) = x - \alpha \int_0^x (x-s) \sin[u(s)] \, ds. \]

This is a Volterra equation. It is not a linear Volterra equation, but that’s OK.

The only means for establishing uniqueness that we have discussed this semester is through the Contraction Mapping Theorem. Let the operator

\[ T[u] = x - \alpha \int_0^x (x-s) \sin[u(s)] \, ds \]

act on the space \( C^2[0,1] \) with the sup norm \( \|f\| = \max_{x \in [0,1]} |f(x)| \). This is a complete metric space where the metric is induced by the norm \( d(u,v) = \|u-v\| \). We must know when \( T \) defines a contraction...
mapping. To do so, we must measure $T[u] - T[v]$.

\[
T[u] - T[v] = \alpha \int_0^x (x - s) \{\sin[v(s)] - \sin[u(s)]\} \, ds
\]

\[
\|T[u] - T[v]\| = |\alpha| \int_0^x (x - s) |\sin[v(s)] - \sin[u(s)]| \, ds
\]

\[
\leq |\alpha| \int_0^x (x - s) \max_{t \in \mathbb{R}} |\cos(t)| |v(s) - u(s)| \, ds
\]

\[
\leq |\alpha| \int_0^x (x - s) |v(s) - u(s)| \, ds
\]

\[
\leq |\alpha| \|v - u\| \int_0^x (x - s) \, ds
\]

\[
\leq \frac{|\alpha|}{2} \|v - u\|
\]

Thus, if $|\alpha| < 2$, $T$ is a contraction mapping and the iterative procedure $u_{n+1} = T[u_n]$ will converge to a unique solution.

2. [25 pts] Find the distributional derivative of

\[
f(x) = H(x)x^{-\frac{3}{2}}.
\]

Is $-\langle f, \phi' \rangle$ a convergent integral? Is the distributional derivative a weak derivative?

To answer the easiest question first: Yes,

\[
\langle f, \phi' \rangle = \int_0^\infty x^{-\frac{3}{2}} \phi'(x) \, dx
\]

cconverges. A number of arguments could be made. I would settle for the one made is class which was loosely based on the Lebesgue Dominated Convergence Theorem. We have not gone over LDCT, but I was happy to see anyone follow the argument given in class. Essentially $x^{-\frac{3}{2}} \psi'(x) \leq Mx^{-\frac{3}{2}}$ where $M$ is a bound on $\psi'$, and the latter is absolutely integrable.

To calculate the distributional derivative, we must avoid boundary contribution at the origin.

\[
-\langle f, \phi' \rangle = -\int_0^\infty x^{-\frac{3}{2}} \phi'(x) \, dx
\]

\[
= -\lim_{\epsilon \to 0} \int_\epsilon^\infty x^{-\frac{3}{2}} \phi'(x) \, dx
\]

\[
= -\lim_{\epsilon \to 0} \left[ e^{-\frac{3}{2}} \phi(\epsilon) - \frac{1}{2} \int_\epsilon^\infty x^{-\frac{3}{2}} \phi(x) \, dx \right]
\]

Both terms inside the limit are divergent as $\epsilon \to 0$, but the limit of the sum exists because we know that $-\langle f, \phi \rangle$ exists. However, this expression is not associated with $\langle g, \phi \rangle$ where $g$ is some locally integrable function, so the distributional derivative is not a weak derivative.
3. [25 pts] Find a Green’s function for the Poisson equation

\[ \nabla^2 u = f(x, y) \]

on the domain \(0 \leq y \leq x < \infty\) with boundary conditions \(u(x, 0) = 0\) and \(u(x, x) = 0\) for \(0 \leq x < \infty\).

This problem is pure fun. Knowing that the Green’s function for the Laplace equation on an unbounded domain is

\[ g(x, y, \xi, \eta) = \frac{1}{4\pi} \ln \left[ (x - \xi)^2 + (y - \eta)^2 \right], \]

we apply the method of images with two planes of symmetry.

To satisfy the BC’s on the x-axis, we would place a negative image solution at position 1 as shown. To satisfy the BC’s at \(y = x\), we would add image solutions at points 2 \((\eta, \xi)\) and 3 \((-\eta, \xi)\) as shown.

Of course, the solutions corresponding to points 2 and 3 do not satisfy the BC’s on the x-axis. Thus, we must add images 4 \((\eta, -\xi)\) and 5 \((-\eta, -\xi)\). Cursed again, we fail to satisfy the symmetry about \(y = x\) with points 4 and 5 so we add 6 \((-\xi, \eta)\) and 7 \((-\xi, -\eta)\). Just when we thought the process might go on forever, we see that images 6 and 7 also satisfy the BC’s on the x-axis.

So, our solution is

\[ g(x, y, \xi, \eta) = \frac{1}{4\pi} \ln \left\{ \frac{[(x - \xi)^2 + (y - \eta)^2] [(x + \eta)^2 + (y - \xi)^2]}{[(x - \xi)^2 + (y + \eta)^2] [(x - \eta)^2 + (y - \xi)^2]} \cdot \frac{[(x - \eta)^2 + (y + \xi)^2]}{[(x + \eta)^2 + (y + \xi)^2]} \right\}. \]
4. [25 pts] Consider the variable conductivity steady heat transport problem

\[-(k(x)u')' = 1\]

with boundary conditions \(u(0) = 0\) and \(u'(1) = 0\), and conductivity

\[k(x) = \begin{cases} 
1, & x < \frac{1}{2} \\
2, & x \geq \frac{1}{2}.
\end{cases}\]

Determine a solution using a suitable Green’s function. Is this solution a classical solution? Why or why not?

First, one must find the Green’s function using the methods discussed in class and in the homework. We worked this problem in the general conductivity \(k(x)\) in class and found that

\[g(x, \xi) = \begin{cases} 
\int_0^x \frac{1}{k(s)} ds, & \xi > x \\
\int_0^\xi \frac{1}{k(s)} ds, & \xi < x.
\end{cases}\]

I did not expect anyone to memorize Green’s function, and so finding it was part of the solution and points were awarded for doing so.

For this specific problem with piecewise constant conductivity, we find that if \(x < 1/2\), we see that \(k\) might vary over the domain of integration.

\[g(x, \xi) = \begin{cases} 
x, & \xi > x \\
\xi, & \xi < x.
\end{cases}\]

However, when \(x > 1/2\),

\[g(x, \xi) = \begin{cases} 
\frac{1}{2}x + \frac{1}{4}, & \xi > x \\
\xi, & \frac{1}{2} < \xi < x \\
\frac{1}{2}\xi + \frac{1}{4}, & \frac{1}{2} < \xi < x.
\end{cases}\]

Using these Green’s functions, we can write out the full solution.

\[u(x) = \int_0^1 g(x, \xi) d\xi = \begin{cases} 
-\frac{1}{2}x^2 + x, & x < \frac{1}{2} \\
-\frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{16}, & x > \frac{1}{2}.
\end{cases}\]

Clearly, both pieces are infinitely smooth except at \(x = 1/2\). At \(x = 1/2\), the second derivative is not continuous, so this cannot be a classical solution. In fact, closer examination will reveal that \(u'\) is not continuous at \(x = 1/2\).
5. [Bonus] Describe general conditions for a source term \( f(x) \) in

\[-(k(x)u')' = f(x)\]

with the conductivity and boundary conditions from problem 4 that would yield classical solutions.

Well, let's consider the solution in the general case for smooth \( f(x) \). Again, we will only have to be concerned at \( x = \frac{1}{2} \) because the Green's function is smooth.

\[
\begin{align*}
  u(x) &= \begin{cases} 
    \int_0^x f(\xi)d\xi + x \int_x^1 f(\xi)d\xi, & x < \frac{1}{2} \\
    \int_0^x \xi f(\xi)d\xi + \int_x^\frac{1}{2} (\frac{1}{2}\xi + \frac{1}{4}) f(\xi)d\xi + (\frac{1}{2}x + \frac{1}{4}) \int_x^1 f(\xi)d\xi, & x > \frac{1}{2}
  \end{cases}
\end{align*}
\]

If \( f \) is smooth, this will be continuous.

Examining the first and second derivatives we find.

\[
\begin{align*}
  u'(x) &= \begin{cases} 
    \frac{1}{x} f(\xi)d\xi, & x < \frac{1}{2} \\
    \frac{1}{2} f(\xi)d\xi, & x > \frac{1}{2}
  \end{cases} \\
  u''(x) &= \begin{cases} 
    -f(x), & x < \frac{1}{2} \\
    -\frac{1}{2} f(x), & x > \frac{1}{2}
  \end{cases}
\end{align*}
\]

Thus, for both derivatives to be continuous at \( x = \frac{1}{2} \), we need \( f(1/2) = 0 \) and \( \int_0^1 f(\xi)d\xi = 0 \).