UNIVERSITY OF DELAWARE
DEPARTMENT OF MATHEMATICAL SCIENCES

Second Midterm
MATH 351
April 2, 2004, 9:05 – 9:55 a.m.

Name: ________________________________ (please print)

family name

given name

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO.

2. Write your name above in block letters.

3. No electronic devices other than scientific and graphing calculators are allowed.

4. This exam has 6 questions on 11 pages — please check to make sure your exam is complete.

5. Note that blank pages have been provided for you to show your work. If the space provided is insufficient you may use the back of the previous page.

6. SHOW INTERMEDIATE STEPS OF YOUR CALCULATIONS.

7. All university rules and guidelines for student conduct are applicable.

8. "My greatest concern is not whether you have failed, but whether you are content with your failure." – Abraham Lincoln
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1. For parts (a)-(d), write the letter(s) that describe the set of vectors.

   A. Linearly independent.
   B. Orthogonal.
   C. Orthonormal.
   D. A spanning set of \( \mathbb{R}^n \), where \( n \) is the number of vectors in the set.

   More than one letter might describe a set.

   [5] (a) \[
   \left\{ \begin{pmatrix} 1 \\ 0 \\ \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ \end{pmatrix} \right\}
   \]

   A, D. To see A, apply the linear independence test: if

   \[
   \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \end{pmatrix}
   \]

   has only the solution \( \alpha_1 = \alpha_2 = 0 \), then the vectors are linearly independent. Forming the augmented matrix and solving using Gaussian elimination quickly shows this to be the case. To see that the vectors are not orthogonal, just dot them together:

   \[
   \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + 0 = 1 \neq 0.
   \]

   Clearly, neither B nor C holds. To see D, form a general vector in \( \mathbb{R}^2 \) as a linear combination of the two vectors:

   \[
   \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.
   \]

   If this has a solution (i.e., \( \alpha_1 \) and \( \alpha_2 \)) for any \( x \) and \( y \), then the vectors do span \( \mathbb{R}^2 \). This is the case here (time-saver: we can see this directly from the row reductions we did for the linear independence check).

   [5] (b) \[
   \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} , \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} , \begin{pmatrix} 3 \\ 6 \\ -5 \end{pmatrix} \right\}
   \]

   A, B, D. Since the vectors are orthogonal (which we confirm by showing that the dot products between any two vectors are all zero), we have to check to see if they’re also normalized to see if C holds. The first vector has magnitude

   \[
   \| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \| = \sqrt{1 + 4 + 9} = \sqrt{14} \neq 1,
   \]

   so this is not an orthonormal set.
None of the letters applies. Note that we can dismiss C immediately since these vectors are in $\mathbb{R}^2$, not $\mathbb{R}^3$!

A, B, C, D.
2. A simple harmonic oscillator is to be constructed with mass $m = 10$ kg and a spring with spring constant $k = 2 \frac{kg}{sec^2}$. Calculate the range of values of the damping coefficient $c$ so that the system will not oscillate.

We would like our system to be critically damped or overdamped to prevent it from oscillating.

If we look for solutions of the form $e^{rt}$ then we want $r$ to be real. For $r$ to be real, we need the discriminant to be positive, i.e. $c^2 \geq 4mk$. Therefore, we need $c^2 \geq 80$. The system will not oscillate if $c \geq \sqrt{80}$. 
3. Above are two sets of three pictures of a harmonic oscillator. The top three pictures are phase portraits (where $y \equiv x'$), and the bottom three pictures are time evolutions of $x(t)$. Match each of the equations below with one of the top three pictures and with one of the bottom three pictures. Explain your reasoning. All oscillators have been started at the initial condition $x(0) = 1$, $x'(0) = 0$.

1. $x'' + x' + 3x = 0$. P3, E3
2. $x'' + 3x = 0$. P1, E2
3. $x'' + 4x' + 3x = 0$. P2, E1
The key step here is identifying what case we have for each oscillator:

1. \( c^2 - 4mk = 1^2 - 4(1)(3) = -11 < 0 \): underdamped

2. \( c = 0 \): undamped

3. \( c^2 - 4mk = 4^2 - 4(1)(3) = 16 - 12 = 4 > 0 \): overdamped

Of the phase portraits, P1 is undamped and oscillates happily with the same amplitude indefinitely. Although P2 and P3 are both damped, P3 undergoes spirals as it decays to zero while P2 does not. P3 must therefore be underdamped, leaving P2 as the overdamped case. Of the evolution pictures, E2 is clearly undamped. E3 undergoes oscillations as it decays to zero; these oscillations correspond to the spirals in P3. E3 is therefore underdamped, while E1 is overdamped.
4. Perform the following operations, leaving your answer in as simple a form as possible.

(a) Expand and simplify

\[(2i + 2)(3 - i)\].

*This is an exercise in algebra knowing that \(i^2 = -1\).*

\[(2i + 2)(3 - i) = 6i + 6 - 2i^2 - 2i = 4i + 8.\]

(b) Expand and simplify

\[(1 - 2i)^3\].

*Again, we just crank it out.*

\[(1 - 2i)^3 = 2i - 11.\]

(c) Rewrite in terms of trigonometric function(s)

\[\frac{1}{4} (e^{ix} + e^{-ix}).\]

*Here, there are a couple of approaches. First, one might recall that*

\[\cos \theta = \frac{1}{2} (e^{ix} + e^{-ix}),\]

*so that*

\[\frac{1}{4} (e^{ix} + e^{-ix}) = \frac{1}{2} \cos x.\]
This is the fastest approach. The second approach requires knowing

\[ \text{This is the fastest approach. The second approach requires knowing} \]

\[ e^{\text{i} \theta} = \cos \theta + \text{i} \sin \theta. \]

**Substituting this identity into the expression, we have**

\[
\frac{1}{4} \left( e^{\text{i}x} + e^{-\text{i}x} \right) = \frac{1}{4} \left[ \cos x + \text{i} \sin x + \cos(-x) + \text{i} \sin(-x) \right]
\]

\[
= \frac{1}{4} \left( \cos x + \text{i} \sin x + \cos x - \text{i} \sin x \right)
\]

\[
= \frac{1}{4} \cdot 2 \cos x = \frac{1}{2} \cos x.
\]
5. Find the solution space of the following system of equations, expressing your answer as the span of one or more vectors.

\[
\begin{align*}
    x_1 - x_2 + x_3 + x_4 &= 0 \\
    2x_1 - x_3 &= 0 \\
    x_1 + 2x_2 + 3x_3 - x_4 &= 0
\end{align*}
\]

Solve using Gaussian elimination:

\[
\begin{pmatrix}
    1 & -1 & 1 & 1 & 0 \\
    2 & 0 & -1 & 0 & 0 \\
    1 & 2 & 3 & -1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & -1 & 1 & 1 & 0 \\
    0 & 2 & -3 & -2 & 0 \\
    1 & 2 & 3 & -1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & -1 & 1 & 1 & 0 \\
    0 & 2 & -3 & -2 & 0 \\
    0 & 0 & 13/2 & 1 & 0
\end{pmatrix}
\]

This has three leading nonzero entries and four variables; we therefore have one parameter to choose. Pick as the parameter the variable that doesn't have a leading entry in its column: let \( x_4 = \alpha \). Then back-substitute:

\[
\begin{align*}
    13x_3 + 2x_2 &= 0 \Rightarrow x_3 = -\frac{2}{13} \alpha \\
    2x_2 - 3x_3 - 2x_4 &= 0 \Rightarrow x_2 = \frac{10}{13} \alpha \\
    x_1 - x_2 + x_3 + x_4 &= 0 \Rightarrow x_1 = -\frac{1}{13} \alpha,
\end{align*}
\]

or writing the solution as a vector,

\[
\mathbf{x} = \begin{pmatrix}
    -\frac{1}{13} \alpha \\
    \frac{10}{13} \alpha \\
    -\frac{2}{13} \alpha \\
    \frac{1}{13} \alpha
\end{pmatrix} = \alpha \begin{pmatrix}
    -1 \\
    10 \\
    -2 \\
    13
\end{pmatrix}.
\]

Since \( \alpha \) is an arbitrary constant, we see that the solution space is

\[
\text{span} \left\{ \begin{pmatrix}
    -1 \\
    10 \\
    -2 \\
    13
\end{pmatrix} \right\}.
\]
6. Find the equation of the plane in $\mathbb{R}^3$ spanned by the vectors
\[
\begin{bmatrix}
-1 \\
1 \\
1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
0 \\
2 \\
-1
\end{bmatrix}.
\]

It's just a matter of doing row operations to the expression
\[
c_1 \begin{bmatrix}
-1 \\
1 \\
1
\end{bmatrix} + c_2 \begin{bmatrix}
0 \\
2 \\
-1
\end{bmatrix} = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]
to constrain $x$, $y$, and $z$.

\[
\begin{bmatrix}
-1 & 0 & x \\
1 & 2 & y \\
1 & -1 & z
\end{bmatrix} \rightarrow \begin{bmatrix}
-1 & 0 & x \\
0 & 2 & y+x \\
0 & -1 & z+x
\end{bmatrix} \rightarrow \begin{bmatrix}
-1 & 0 & x \\
0 & 2 & y+x \\
0 & 0 & z+x+\frac{1}{2}(y+x)
\end{bmatrix} \rightarrow \begin{bmatrix}
-1 & 0 & x \\
0 & 2 & y+x \\
0 & 0 & z+\frac{3}{2}x+\frac{1}{2}y
\end{bmatrix}
\]

Thus, the equation of the plane is
\[
\frac{3}{2}x + \frac{1}{2}y + z = 0.
\]