Exam 3 solution guide
21 November 2003

Mathematics may claim to be the most original creation of the human spirit.
—Alfred Whitehead

Instructions: Show all work to receive full or partial credit. You may use Maple and
a scientific calculator on this exam. All University rules and guidelines for student conduct
are applicable.

1. [20 pts] Find the maximum and minimum values of
\[ f(x, y, z) = x^2y^2z^2 \]
constrained to the ellipsoid
\[ x^2 + y^2 + \frac{z^2}{4} = 1. \]

28.6% of the class earned 80% or more of the points on this question. This problem is
similar to questions 3-17, especially 10, in your textbook section 15.8.

We will use Lagrange multipliers and solve the problem
\[ \nabla f = \lambda \nabla g \]
\[ \nabla f = 2(xy^2z^2, x^2yz^2, x^2y^2z) \]
\[ \nabla g = (2x, 2y, z/2) \]

Thus, we must solve the system
\[ xy^2z^2 = \lambda x \]
\[ x^2yz^2 = \lambda y \]
\[ x^2y^2z = \lambda z/4 \]

If we assume that none of \( x, y \) and \( z \) are not zero then
\[ y^2z^2 = x^2z^2 = 4x^2y^2. \]

We get this by solving for \( \lambda \) in the three equations above. Therefore,
\[ x^2 = y^2 = z^2/4. \]

If we substitute these expressions into the constraint, we have that
\[ x = \pm\sqrt{1/3}, \quad y = \pm\sqrt{1/3}, \quad z = \pm2/\sqrt{3}. \]
At any of these points, \( f(x, y, z) = \frac{4}{27} \).

However, we can see that any of \( x, y \) or \( z \) could be zero and eliminate one or more of the original equations relating \( x, y, z \) and \( \lambda \). Clearly, \((0,0,0)\) does not satisfy the constraint. Assuming that at least one of \( x, y \) or \( z \) is nonzero, this would necessarily imply that \( \lambda = 0 \). For instance, the points \((\pm 1, 0, 0)\) are critical points. However, since one of either \( x, y \) or \( z \) is zero, \( f(x, y, z) = 0 \) at these points where \( \lambda = 0 \).

Therefore, the maximum value is \( \frac{4}{27} \) and the minimum is 0.
2. [10 pts] Set up **but do not evaluate** \( \int \int_R f(x, y) \, dA \) where \( R \) is the shaded region shown below.

\[
\begin{align*}
\int_{-1}^{1} & \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) \, dy \, dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2/4}} f(x, y) \, dy \, dx \\
& + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) \, dy \, dx + \int_{1}^{2} \int_{\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} f(x, y) \, dy \, dx
\end{align*}
\]

57.1% of the class earned 80% or more of the points on this question. This problem is similar to questions 7-18 in your textbook section 16.3.

The more direct way to set this up is to integrate with \( x \) for the inner integral.

\[
\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) \, dx \, dy + \int_{-1}^{1} \int_{\sqrt{1-y^2}}^{\sqrt{2-1-y^2}} f(x, y) \, dx \, dy
\]

3. [10 pts] Reverse the order of integration for the integral you set up in problem 2.

28.6% of the class earned 80% or more of the points on this question. This problem is similar to questions 33-44 in your textbook section 16.3.
4. [20 pts] Find the mass of the solid bounded by the two paraboloids

\[ y = x^2 + z^2, \]
\[ y = 4 - 2x^2 - 2z^2, \]

if the density of the solid is \( \sigma = y \).

4.8% of the class earned 80% or more of the points on this question. This problem is similar to questions 7-20 in your textbook section 16.7.

There are many ways to set this problem up and calculate the mass. One expedient way to do it is with cylindrical coordinates but treating \( y \) as the vertical direction. Thus, \( x = r \cos \theta, z = r \sin \theta \) (so \( x^2 + z^2 = r^2 \)), and \( y = y \). Thus, in the new coordinates, the equations for the surfaces are

\[ y = r^2, \]
\[ y = 4 - 2r^2. \]

The two surfaces intersect when \( r = 2/\sqrt{3} \). Thus, the mass of the solid is

\[
m = \int_{0}^{2\pi} \int_{0}^{2/\sqrt{3}} \int_{r^2}^{4-2r^2} \sigma \, dr \, dy \, d\theta = \int_{0}^{2\pi} \int_{0}^{2/\sqrt{3}} \int_{r^2}^{4-2r^2} y \, dr \, dy \, d\theta = \frac{384}{81} \pi = \frac{128}{27} \pi.
\]
5. [20 pts] Evaluate $\iiint_R xyzdV$ where $R$ is bounded between the sphere of radius 1 and the sphere of radius 4, and above the cone $\phi = \frac{\pi}{4}$.

38.1% of the class earned 80% or more of the points on this question. This problem is similar to questions 17-30, especially 22, in your textbook section 16.8.

This is a direct application of triple integration in spherical coordinations because the endpoints are constants in spherical coordinations. Some noticed right off the bat that this integral involves integration of an odd function (in both $x$, $y$ and $z$) over symmetric intervals, so we expect this to be zero.

$$\iiint_R xyzdV = \int_0^{2\pi} \int_0^{\pi/4} \int_1^4 xyz\rho^2 \sin \phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_1^4 (\rho \cos \theta \sin \phi)(\rho \sin \theta \sin \phi)(\rho \cos \phi)\rho^2 \sin \phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_1^4 \rho^5 \cos \theta \sin \theta \cos \phi \sin^3 \phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{6} (4^6 - 1) \cos \theta \sin \theta \cos \phi \sin^3 \phi d\rho d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{6} (4^6 - 1) \frac{1}{2} \sin^2 \theta |_0^{2\pi} \cos \phi \sin^3 \phi d\phi$$

$$= 0.$$
6. [20 pts] Find $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y) = \langle x^2, y^3 \rangle,$$

and $C$ is the line segment from $(0, 1)$ to $(1, 0)$.

81.0% of the class earned 80% or more of the points on this question. This problem is similar to questions 19-22 in your textbook section 17.2.

We can solve the problem directly by finding a parametrization for the line segment. One such parametrization would be

$$\mathbf{r}(t) = \langle 0, 1 \rangle + \langle 1, -1 \rangle t, \quad t : 0 \rightarrow 1.$$  

Thus, $\mathbf{r}'(t) = \langle 1, -1 \rangle$.

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(t, (1 - t)) \cdot \mathbf{r}'(t) dt$$

$$= \int_0^1 \langle t^2, (1 - t)^3 \rangle \cdot \langle 1, -1 \rangle dt$$

$$= \frac{1}{12}.$$

OK. Now, there is a much easier way to go. If you test the vector field, we see that it has a potential,

$$f(x, y) = \frac{1}{3} x^3 + \frac{1}{4} y^4.$$

Then, $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = f(1, 0) - f(1, 0) = \frac{1}{12}.$