1. On the same graph, generate the stability regions for the following two method Adams-Bashforth-Moulton methods.

(a) Fourth order Adams-Moulton using a third order Adams-Bashforth predictor.

First, we recall that AB3 is

\[ u_{k+1} = u_k + \frac{h}{12} (23f_k - 16f_{k-1} + 5f_{k-2}) , \]

and AM4 is

\[ u_{k+1} = u_k + \frac{h}{24} (9f_{k+1} + 19f_k - 5f_{k-1} + f_{k-2}) . \]

If we solve the scalar test equation, and use AB3 as a predictor for AM4, we see that the \( h\lambda \) satisfies,

\[ (h\lambda)^2 a(r) + (h\lambda)b(r) + c(r) = 0 , \]

where

\[ a(r) = \frac{1}{32} (23r^2 - 16r + 5) , \]
\[ b(r) = -\frac{1}{24} (28r^3 - 5r + 1) , \]
\[ c(r) = r^2 - r^3 . \]

If you plot the real and imaginary parts of the solutions for \( h\lambda \), you generate the following stability plot.

![Stability Plot](image-url)
(b) Fourth order Adams-Moulton using a fourth order Adams-Bashforuth predictor.

This problem is very similar to the previous one. We recall that \( AB_4 \) is

\[
u_{k+1} = u_k + \frac{h}{24} (55f_k - 59f_{k-1} + 37f_{k-2} - 9f_{k-3}).
\]

With this predictor, we find that

\[
a(r) = \frac{1}{64} (55r^3 - 59r^2 + 37r - 9),
\]

\[
b(r) = -\frac{1}{24} (28r^3 - 5r^2 + r),
\]

\[
c(r) = r^3 - r^4.
\]

With these coefficients, we find the stability region to be

2. [KC 8.3:5] Derive the third-order Runge-Kutta formulas

\[
u_{k+1} = u_k + \frac{1}{9} (2F_1 + 3F_2 + 4F_3)
\]

where

\[
F_1 = hf(u_n, t)
\]

\[
F_2 = hf \left(u_n + \frac{1}{2} F_1, t + \frac{1}{2} h \right)
\]

\[
F_3 = hf \left(u_n + \frac{3}{4} F_2, t + \frac{3}{4} h \right)
\]

Show that it agrees with the Taylor-series method of order 3 for the differential equation

\[y' = y + t.\]
This problem is just a matter of substituting \( f(y, t) = y + t \) into the expressions for \( F_1 \), \( F_2 \) and \( F_3 \), and simplifying.

3. [KC 8.3:6] Prove that the when the fourth order Runge-Kutta method is applied to the problem

\[
y' = -\lambda y,
\]

the formula for advancing this solution will be

\[
u_{k+1} = \left[1 - h\lambda + \frac{1}{2}(h\lambda)^2 - \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right]u_k.
\]

Again, this is just a matter of substituting \( f(y, t) = -\lambda y \) into RK4.

4. [KC 8.3:7]

Prove that the local truncation error in the previous problem is \( O(h^5) \).

To find the local truncation here, just look at the next term in the Taylor series of \( e^{-\lambda t} \). This is the first term where RK4 and the exact solution disagree, so the local truncation error is \( \frac{1}{5!}(h\lambda)^5 \) which is \( O(h^5) \).

5. [Mini project - double credit]

An idealized aircraft wing can be modeled as a point vortex in free space which induces a flow field

\[
\mathbf{u}(x, y; \Gamma, x_0, y_0) = \frac{\Gamma}{2\pi} \frac{1}{(x-x_0)^2 + (y-y_0)^2} \begin{bmatrix} - (y - y_0) \\ (x - x_0) \end{bmatrix}
\]

(1)

where \((x_0, y_0)\) is the position of the wing, and \(\Gamma\) is the circulation which is proportional to aerodynamic lift.\(^1\) In steady flight wherein the aircraft experiences no accelerations, the aerodynamic lift balances the weight of the aircraft, and the thrust produced by the engine balances aerodynamic drag.

Suppose the winds are calm, and an aircraft is on final approach to a runway. The aircraft is trimmed so that it is descending at a rate of 50 feet per minute (fpm) and moving forward at 4000 fpm. With the current trim settings (flaps, elevator, pitch, etc), \(\Gamma = 10^5\). The landing gear is located 6 feet below the wing.

At \( t = 0 \) minutes, the wing is 56 feet above ground and 4000 feet from the touchdown zone. The pilot, oblivious to “ground effect”, thinks the aircraft will land in the

\(^1\)Notice that the flow field is singular at the wing in this idealization.
touchdown zone at $t = 1$ minutes. The pilot is right about the time but wrong about where the plane will land!

The free space flow field (1) is not applicable at low altitude because air cannot move in or out of the ground. To satisfy this boundary condition, we must add a virtual image wing that appears to fly underground and cancels the vertical component of the velocity field at ground level. Thus, the true flow field, $\mathbf{v}$ is

$$\mathbf{v}(x, y) = \mathbf{u}(x, y; \Gamma, x_0, y_0) + \mathbf{u}(x, y; -\Gamma, x_0, -y_0),$$

and the aircraft will be pushed forward by the flow field induced by the virtual wing.

(a) You do not need numerical analysis to compute the position of the aircraft as a function of time. Plot the trajectory of the aircraft for $0 \leq t \leq 1$. How far beyond the touchdown zone does the aircraft land?

We know that the vertical position of the wing is going to be

$$y_0 = 56 - t.$$

First, we must solve the ODE corresponding to the affect of the image vortex on the wing.

$$x_0' = 4000 + \frac{10^5}{2\pi} \frac{1}{2(56 - 50t)}$$

If we integrate this function assuming that $x_0(0) = -4000$, we find that

$$x_0 = 4000t - \frac{10^5}{200\pi} \ln \left( \frac{56 - 50t}{56} \right) - 4000.$$

To answer the question, we find that

$$x_0(1) = \frac{500}{\pi} \ln \left( \frac{28}{3} \right) \approx 355\text{ft}.$$

(b) Eli is crouching in the touchdown zone smoking a cigarette. How low must he crouch to avoid getting hit by the landing gear? You may assume that the cigarette is 2.5 feet off the ground. Compute the smoke made by his cigarette smoke assuming there is no diffusion. You can do this by releasing a particle of smoke and allowing it to move with the flow velocity induced by the aircraft and its image wing. If you release particles every second or fraction of a second, the cloud of particles will generate the pattern.

Eli is safe as long as he ducks. Looking at $x_0$ and $y_0$ as described above, the landing gear will pass about 3.5 over the touchdown zone. Here is a picture of Eli's smoke pattern. The red cross is the wing position at time 0.9 minutes and 1 minute. The green lines are the wind velocity vectors. The blue spots are the puffs of smoke. Eli releases one puff of smoke every $\frac{1}{100}$ of a second.