1. Derive the optimal coefficients for 3rd order Adams-Bashforth.

So, we are seeking the coefficients $A_1, A_2, A_3$ for the AB3 method

$$u_{k+1} = u_k + h(A_1 f_k + A_2 f_{k-1} + A_3 f_{k-2}).$$

The procedure is straightforward. Remember, the coefficients are independent of $h$ so what works for one value of $h$ works for all values of $h$. Let $h = 1$ to keep it simple. For a third order method, we expect to choose coefficients such that the $u_k$'s are exact for any cubic polynomial. So, we shall choose a test function

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3,$$

and so,

$$y' = c_1 + 2c_2 x + 3c_3 x^2.$$

Substituting this into the method, we have

$$c_0 + c_1 + c_2 + c_3 = c_0 + A_1 c_1 + A_2 (c_1 - 2c_2 + 3c_3) + A_3 (c_1 - 4c_2 + 12c_3).$$

This is true for all cubics, so

$$c_1 (1 - A_1 - A_2 - A_3) + c_2 (1 + 2A_2 + 4A_3) + c_3 (1 - 3A_2 - 12A_3) = 0$$

and

$$1 - A_1 - A_2 - A_3 = 0$$
$$1 + 2A_2 + 4A_3 = 0$$
$$1 - 3A_2 - 12A_3 = 0.$$

When you solve this linear system, you find that $A_1 = \frac{23}{12}, A_2 = -\frac{4}{3}$, and $A_3 = \frac{5}{12}$.

2. [KC 8.4:4] Use the method of undetermined coefficients to derive the fourth-order Adams-Bashforth formula

$$x_{n+1} = x_n + \frac{h}{24} [24f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$$

Just set $h = 1$, plug in a quartic and solve the resulting linear system.
3. [KC 8.4:5] Derive the fourth-order Adams-Moulton formula

\[ x_{n+1} = x_n + \frac{h}{24} [9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] \]

Just set \( h = 1 \), plug in a quartic and solve the resulting linear system.

4. Consider the “poor man’s orbital problem”

\[ x' = -y, \quad y' = x \quad x(0) = 1, y(0) = 0. \]

Solve the problem for \( 0 \leq t \leq 8\pi \) (four revolutions) with \( h = \pi/20 \) and \( h = \pi/40 \) using forward Euler, backward Euler, 3rd order Adams-Bashforth, 3rd order Adams-Bashforth-Moulton. Use an appropriate Taylor series start-up procedure.

I implemented all four methods and plot the results below.
Of course, the Euler methods are not as accurate as the Adams family methods. More interesting is the qualitative behavior of these methods. All methods orbit roughly four times. Notice that the forward method spirals out which is precisely what one would expect from the stability test via the scalar test equation. Remember that the imaginary axis for the forward Euler equation is entirely in the unstable region. The opposite is true for the backward Euler method. The imaginary axis is entirely within the stable region. Ideally, a numerical method would “hug” the imaginary axis in the \( h\lambda \) plane so reproduce oscillator behavior with as little decay or growth as possible.

Looking at the lower two plots, we see that the Adam’s family methods are much more accurate for the given step-size, and they have been stability properties. Remember in class how the AB4 hugged the imaginary axis for small \( h \)? This is important! Both AB3 and ABM3 have the same order of accuracy, but if you examine the stability diagrams ABM3 does a far better job of hugging the imaginary axis than AB3. Below, I zoom in on the starting and ending positions of ABM3. Where AB3 spirals inward slightly, ABM3 spirals outward slightly (why?).