Solution notes
Problem Set #4
Modeling classroom activity #2
One dimensional diffusion models

Every one of these problems explores modeling using a conservation law, but each problem has a different twist. Since \( q \), the CO\(_2\) content, is conserved, the flux is constant across any position \( x \) when the system is in a steady state configuration.\(^1\) Thus, for any of these problems, we must correctly determine an expression for the flux and extend this information into a full solution. As we discussed in class, we can only hope to find an equilibrium if plant absorbs as much CO\(_2\) as the astronaut produces. We shall call this quantity \( F_0 \), and we shall use this quantity at flux boundary conditions as each end of the domain.

1. An astronaut stands at the left end of a long hallway of length \( L \) and produces carbon dioxide at a steady rate. At the other end of a hallway, a large houseplant consumes carbon dioxide at the same rate. Assuming a “Fourier law” diffusion mechanism, develop a one-dimensional mathematical model for the amount of CO\(_2\) as a function of space and time if there is a ventilation system that circulates air from left to right at a speed \( U \).

\[
\begin{array}{c}
\text{U} \\
\downarrow \\
\text{L} \\
\downarrow \\
\text{U}
\end{array}
\]

Find a steady state solution.

This particular problem is just like the example in class except that that the CO\(_2\) is being blown across the domain. Thus, one can hypothesize that there are two transport terms. One is diffusive transport governed by the usual “Fickian” or “Fourier Law” term \(-kq_x\).\(^2\) The other flux contribution would be due to the ventilation system. One can consider each contribution separately, so to understand the affect of the ventilation system, one can set \( k = 0 \) to remove diffusion from the problem. If the distribution at time \( t = 0 \) is \( q(x, 0) \), the distribution at some later time would be \( q(x, t) = q(x-Ut, 0) \). In other words, if one wanted to know \( q \) at any time, one just looks upstream toward

\(^1\)This is only true for the steady-state configuration. If the problem were dynamic, the flux across a position may vary as the system settles toward an equilibrium configuration.

\(^2\)The true diffusive flux would be \(-khq_x \) where \( h \) is the height of the hallway, but the height of the hallway is constant so we can fold this into the parameter \( k \). In problem #2, the height changes, so we need to keep track of \( h(x) \).
the fan. As usual, we can relate the change in \( q \) in a control volume with the flux as follows:

\[
\frac{d}{dt} \int_a^b q(x, t) \, dx = F(a, t) - F(b, t)
\]

where \( a \) and \( b \) are arbitrary. Replacing \( q(x, t) \) with \( q(x - Ut, 0) \), we see that

\[
\frac{\partial}{\partial t} q(x, t) = \frac{\partial}{\partial t} q(x - Ut, 0) = -Uq_x(x - Ut, 0) = -Uq_x(x, t).
\]

Differentiating (1), we find that

\[
\frac{d}{dt} \int_a^b q(x, t) \, dx = F(a, t) - F(b, t)
\]

\[-U \int_a^b q_x(x, t) \, dx = F(a, t) - F(b, t)
\]

\[Uq(a, t) - Uq(b, t) = F(a, t) - F(b, t)
\]

which tells us that the contribution to the flux from the ventilation is \( Uq \). Thus, we see that the total flux is

\[-kq_x + Uq.
\]

Thus, the dynamic system would be

\[q_t = (kq_x - Uq)_x.
\]

The boundary conditions are

\[-kq_x(a, t) + Uq(a, t) = F_0, \quad -kq_x(b, t) + Uq(b, t) = F_0,
\]

and we assume there is some initial distribution \( q(x, 0) = q_0(x) \).

To find a steady state solution, \( q(x) \), we set \( q_t = 0 \) and solve

\[(kq_x - Uq)_x = 0.
\]

Integrating once, and then solving the resulting first-order inhomogeneous equation, we find that

\[q(x) = \frac{F_0}{U} + Ce^{\frac{Ux}{k}}.
\]

Notice that the first constant of integration, \( F_0 \), is the known flux from the boundary conditions. The constant \( C \) is determined by matching the total \( q(x) \) from the steady-state solution:

\[\int_0^L q(x) \, dx = \int_0^L q_0(x) \, dx = Q_0.
\]

Thus,

\[C = \frac{(UQ_0 - F_0L)}{k \left( e^{\frac{UL}{k}} - 1 \right)}.
\]
and

\[
q(x) = \frac{F_0}{U} + \frac{(UQ_0 - F_0L)}{k \left( e^{\frac{U(L-x)}{k}} - e^{-\frac{U}{k}} \right)}
\]

\[
= \frac{F_0}{U} + \frac{(UQ_0 - F_0L)}{2k \sinh \left( \frac{UL}{2k} \right)} e^{\frac{U(L/2)}{k}}
\]

2. Consider the same problem as above without the fan but with a sloping roof in the hallway.

Find a steady state solution.

*Here, the flux term is \(-kh(x)q_x\) as previously discussed. Thus, we must solve*

\[(kh(x)q_x)_x = 0\]

*where*

\[h(x) = \frac{h_2 - h_1}{L} x + h_1.\]

*Integrating the conservation equation once and then solving the differential equation, one finds that*

\[q(x) = -\frac{F_0L}{k(h_2 - h_1)} \ln \left( \frac{h_2 - h_1}{L} x + h_1 \right) + C.\]

*Again, \(F_0\) comes from the known flux at the boundaries and \(C\) must be determined from the total initial amount, \(Q_0\). Hacking through the integral of the solution,*

\[C = \frac{1}{L} \left\{ Q_0 + F_0 \left( \frac{L}{h_2 - h_1} \right)^2 [h_2 (\ln h_2 - 1) - h_1 (\ln h_1 - 1)] \right\}.
\]
3. Consider the same problem, but with a stepped hallway as shown. You may assume that the hallway makes the step at $x = L/2$.

\[
\begin{array}{c}
\text{h1} \\
\text{L} \\
\text{h2}
\end{array}
\]

Find a steady state solution.

It is possible to approach this problem a number of ways including using a Heaviside step function or using a thin transition region from one hallway height to the other and letting the width of the transition go to zero. However, one of the most direct ways to attack this problem is to decompose this problem into two domains, and make each solution compatible with suitable boundary conditions at the new interior boundary at $x = L/2$.

Each domain is a problem just like the one we solved in class, and the solution in each half of the hallway is linear. If we call the solution on the left half, $q_L(x)$, and the solution on the right, $q_R(x)$, we see that

\[
q_L(x) = -\frac{F_0}{kh_1}x + a \quad q_R(x) = -\frac{F_0}{kh_2}x + b.
\]

But, $a$ and $b$ are not known from the boundary conditions at either end. However, for $q$ to be continuous at $x = L/2$, we see that

\[
\frac{F_0 L}{2k} \left( \frac{1}{h_2} - \frac{1}{h_1} \right) = b - a.
\]

Another constraint on the interior boundary would be that the flux would be the same from the left and right, but this adds no new information because we have already imposed a constant flux condition. However, the initial data will give us one more equation with which we can solve this problem. Integrating the piecewise solution and matching it to $Q_0$, we have

\[
\frac{F_0 L}{2k} \left( \frac{1}{h_2} + \frac{1}{h_1} \right) \frac{L^2}{4} + (a + b) \frac{L}{2} = Q_0.
\]

Solving the two linear equations, we find the solution to be

\[
Q_L(x) = -\frac{F_0}{kh_1}x + \frac{Q_0}{L} + \frac{3L}{8} \frac{F_0}{kh_1} + \frac{L}{8} \frac{F_0}{kh_2},
\]

\[
Q_R(x) = -\frac{F_0}{kh_2}x + \frac{Q_0}{L} - \frac{L}{8} \frac{F_0}{kh_1} + \frac{5L}{8} \frac{F_0}{kh_2}.
\]
To conclude, I would like to end with a few comparisons. Let’s standardize the problem so that \( Q_0 = F_0 = L = k = 1 \) and vary the other parameters. Notice the similarities in

![Figure 1: Solutions to problem 1 (red) with \( U = 2 \), and problems 2 (blue) and 3 (green) with \( h_1 = 1/2 \) and \( h_2 = 3/2 \).](image)

the solutions for problems 2 and 3. The relatively high ventilation velocity forces the \( CO_2 \) to pile up at the right side. In this set of solutions, I allowed the slant on the roof to run in the other direction. The relatively lower ventilation velocity means that diffusion is the dominant term in the flux and \( CO_0 \) will accumulate closer to the astronaut.
Figure 2: Solutions to problem 1 (red) with $U = 1/2$, and problems 2 (blue) and 3 (green) with $h_1 = 3/2$ and $h_2 = 1/2$. 