PROJECT GUIDELINES - MATH 810
PERTURBATION ANALYSIS OF LAPLACE’S EQUATION

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Introduction

Asymptotic and perturbation methods were developed for applications. Poincare, one of perturbation method’s pioneers, developed “stretched-variable” techniques in order to better understand planetary motion. Latter researchers such as Prandtl and Cole further developed perturbation methods in studies of fluid mechanics. Today, researchers from every branch of science and engineering rely on asymptotic and perturbation methods when trying to obtain insight into the physics of various phenomena. This strong bond between perturbation methods and applications makes learning perturbation methods in the context of applications a necessity. In this project you will carry out a study of a physical system from multiple points of view. You will setup and perform an experiment in the laboratory, construct a mathematical model of your physical system, scale your model and identify a small parameter, analyze your model using perturbation and computational methods, and compare the results of experiment and analysis.

Our Motivation and Physical System

By now you have undoubtedly encountered the Laplace equation (1) on numerous occasions

\[ \nabla^2 \psi = 0. \] (1)

You may know it from your studies of steady-state heat flow, fluid flow, electrostatics, diffusion, or elasticity. You have probably solved the Laplace equation in simple geometries; the square, the rectangle, the strip, the circle. You may have even solved coupled Laplace equations where material properties varied from one domain to a neighboring domain. A casual glance at the real world is enough to reveal that these simplified situations rarely occur in nature or even in engineered systems. What can we do when the geometry of our system becomes complex? One answer is to turn to numerical computation. A second is to exploit perturbation methods. In this project you will carry out and compare both of these approaches.

While Laplace’s equation arises in a variety of contexts, we will focus on applications to electrostatics. This choice is based on the experimental equipment we have available and on the importance of this application. In modern materials science a tremendous amount of effort goes into creating composite materials with tailored electrical properties. Understanding how the electrical properties vary as a function of material composition leads directly to the Laplace equation and electrostatics.

To understand this, consider the system shown in Figure 1. In general we are interested in the case where the slab of material is a composite, i.e., its electrical properties are a function of position within the slab. The first question we would like to answer is what the electric field inside the slab looks like when a potential difference is applied across the faces of the slab. Recall the Maxwell equations

\[ \nabla \cdot D = \frac{\rho}{\varepsilon_0} \] (2)

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where \( \sigma \) will be. Using Ohm’s law and the expression for \( J \) to the electric field \( E \), Ohm’s law relates the current \( J \) to the electric field \( E \)

\[
\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \tag{3}
\]

\[
\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{4}
\]

\[
\nabla \cdot \mathbf{B} = 0. \tag{5}
\]

Assuming static fields we may set all time derivatives to zero. Equation (4) becomes

\[
\nabla \times \mathbf{E} = \mathbf{0}. \tag{6}
\]

If a vector field is curl free, it may be written as the gradient of a potential. That is, we can write

\[
\mathbf{E} = -\nabla \psi. \tag{7}
\]

Equation (3) becomes

\[
\nabla \times \mathbf{H} = \mathbf{J}. \tag{8}
\]

Ohm’s law relates the current \( \mathbf{J} \) to the electric field \( \mathbf{E} \)

\[
\mathbf{J} = \sigma \mathbf{E} \tag{9}
\]

where \( \sigma \) is the electrical conductivity of the medium. Note that \( \sigma \) may be a function of space. For a composite material, it certainly will be. Using Ohm’s law and the expression for \( \mathbf{E} \) in terms of \( \psi \) yields

\[
\nabla \times \mathbf{H} = \sigma \mathbf{E} \tag{10}
\]

or

\[
\nabla \times \mathbf{H} = -\sigma \nabla \psi. \tag{11}
\]

Taking the divergence across this equation yields

\[
\nabla \cdot (\sigma \nabla \psi) = 0. \tag{12}
\]

If \( \sigma \) is independent of space this reduces to the familiar Laplace equation. Your main goal in this project is to investigate equation (12) when \( \sigma \) is a function of space and when \( \sigma \) is constant, but the geometry of your material is complicated.

### Part One - Goals for February 24th

In the first part of this project, which you should complete by February 24th, you should accomplish the following:

1. Review the derivation of the Laplace equation for electrostatics. Make sure you understand the derivation above. Details have been skipped, fill them in! No discussion of boundary conditions appears above, that’s for you to figure out!
2. Discuss how the Laplace equation arises in heat transfer, fluid flow, elasticity etc. You should derive equation (12) for at least one of these other physical situations. In your final report a thorough discussion of how your work applies to these other situations is needed.
3. Solve the Laplace equation for a rectangular strip with fixed potential difference across two boundaries. Solve analytically and numerically, plot. This is a warm-up problem, look at this as a chance for you to refresh your memory of PDE’s, the Laplace equation, numerical techniques, etc.
4. Perform experiment on rectangular strip, compare with analytic, numerical solutions. This is a warm-up lab based problem. You should use this opportunity to familiarize yourself with the experimental equipment (described below), its shortcomings etc. Also, you should focus on how to compare your theory with experiment. How do you compare data? How close is close enough?

Please view these goals as a starting point for exploration rather than a laundry list of tasks to complete. Your group should spend some time discussing the issues above, discussing approaches, results, successes, and failures.

### Part Two - Goals for March 10th

In the second part of this project, which you should complete by March 10th, you’ll investigate the Laplace equation on an irregular domain. You should accomplish the following:

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1. Find the exact solution of the Laplace equation in an annulus with a potential difference between boundaries. See Figure 2.

2. Setup the Laplace equation for “skewed” annulus. Scale, identify small parameter, develop perturbation theory. I’ve left the idea of “skewed” a bit vague. By now, you should be familiar with the experimental equipment. You want your skewed-theory to be doable experimentally! Some ideas are shown in Figure 2.

3. Solve both the annulus and skewed annulus numerically. I suggest you use the PDE toolbox available in Matlab and described below.

4. Perform experiment on annulus and skewed annulus, compare with analytical, numerical results.

Part Three - Goals for March 24th

In the third part of this project you’ll investigate the Laplace equation for a different sort of irregular geometry. You should complete this work by March 24th.

1. Setup Laplace equation for “rectangular strip” with two wavy edges. Scale, identify small parameter, develop perturbation theory. Again, you need to determine what “wavy” should mean. What can you do numerically? experimentally? Some ideas are shown in Figure 3.

2. Solve numerically.

3. Perform experiment, compare.

Part Four - Goals for April 28th

In the final part of this project you’ll investigate the Laplace equation for a material with spatially varying electrical conductivity. I’ve left the formulation of this system very vague. A challenge for your group is to decide on a problem that is both interesting and that you can do!

1. Decide on model of varying material to investigate

2. Setup Laplace equation, scale, identify small parameter, develop perturbation theory.

3. Perform experiment, compare.

Experimental Equipment and Some Tips

In this project you will make use of Pasco’s field mapper kit. The manual for this kit is available on my web page. The essential component is “resistance paper.” This is a conducting sheet of paper. On this paper you may draw metallic “perfectly conducting” lines. Then, using a power supply you can apply a potential difference across regions of the paper. The electric field inside these regions satisfies Laplace’s equation! You can measure the field using a voltmeter. I suggest you stop by the lab as soon as possible to familiarize yourself with this equipment. Don’t just come by to look - come by to play. How well does this work? How well does it match up with what you expect from Laplace? How can you get “good data?” These are all questions you’ll need to answer. Some tips/observations:

1. Pay close attention to the conducting ink, is it as conducting as you think?

2. The paper is paper! You can cut it, punch holes in it, press metal against it, fold it, and write on it.

Numerical Tools

As part of this project you are asked to compare your perturbation and experimental results with the results of numerical simulations. Fortunately, accurate easy-to-use numerical packages are available that handle the Laplace equation in a wide variety of situations. You are free to use whatever numerical package you wish. You can, if so-motivated, write your own code. I suggest that you use the PDE Toolbox available in Matlab. This toolbox has a graphical user interface that allows you to sketch domains, create meshes, specify boundary conditions, solve PDE’s, and plot the solutions. This can all be accomplished with a minimum of effort. To start the graphical user interface simply type “pdetool” at Matlab’s command line. A sample solution of Laplace’s equation in a domain with holes is shown in Figure 4. From start to finish generating this solution and figure took less than 10 minutes.
Figure 4. Sample solution of Laplace using Matlab.

Report Guidelines

As the semester progresses I will update these guidelines, especially this final section. Some preliminary guidelines:

1. A first draft of your final report is due May 12th. I will read and comment on these drafts. I expect you to make significant changes based on my comments.
2. Your report should be structured as a journal paper. The level of detail should be slightly higher than a journal paper. In particular, I want to see the details of your perturbation analysis.
3. Your report should be understandable to the readership of the SIAM Journal on Applied Mathematics. That is, do not think of me as your target reader, think of a more general audience.
4. You should include figures and plots as needed.
5. You should use Latex to prepare your report. (That is a suggestion.)
6. At the end of the class, you need to turn in a paper copy of your report and a PDF version.

I really didn’t have much to teach. I didn’t even believe in it. I felt so strongly that everybody has to find their own way. And nobody can teach you your own way...

In terms of art, the only real answer that I know of is to do it. If you don’t do it, you don’t know what might happen.
- Harry Callahan, 1991