Assignment 3  
Math 810 - Spring 2003  
Prof. John A. Pelesko

(1) Consider the perturbed eigenvalue problem

\[ y'' + \lambda (1 + \epsilon x) y = 0 \]

\[ y(0) = y(\pi) = 0. \]

Assume \( \epsilon \ll 1 \) and find a leading order approximation to the eigenfunctions and the first two terms in the expansion of the eigenvalues. (Hint: You need to expand both \( y \) and \( \lambda \) and use the Fredholm alternative theorem.)

(2) Consider the problem of steady-state heat conduction in a rod of length \( L \)

\[ \kappa \frac{d^2T}{dx^2} + s(x') = 0 \]

where here, \( \kappa \) is the thermal conductivity and \( s(x') \) is a source term. Assume that \( s(x') = p \) for \( 0 \leq x' \leq x_0 \) and is identically zero for \( x_0 < x' < L \). As boundary conditions, assume that the rod is insulated on the left so that \( T'(0) = 0 \) and is in an ice bath on the right so that \( T(L) = T_b \).

(a) Introduce dimensionless variables and scale your equations.

(b) Assuming a strong source argue that a dimensionless parameter from part (a) is small.

(c) Using boundary layer theory find a uniform approximation to the solution.

(3) Consider the nonlinear oscillator equation with a “soft” nonlinearity

\[ u'' + u - \epsilon u^3 = 0 \]

Impose initial conditions

\[ u(0) = 0 \]

\[ u'(0) = a \]

(a) Using an energy argument, derive a potential for this problem. Sketch the potential well. What conditions must you impose on \( a \) in order to guarantee that you have a bounded (and hence periodic) solution?

(b) Assume \( \epsilon \ll 1 \) and expand the solution to this problem in a regular perturbation series. Compute at least the first two terms. Is your expansion uniformly valid? Why or why not?

(c) Assume \( \epsilon \ll 1 \) and apply the Poincare-Lindstedt method to this problem. Determine at least the first correction to the frequency.

(d) Solve this problem numerically and compare the approximations obtained in parts (b) and (c) with a numerical solution. Discuss your findings.