1. (Scale changes). Show that if $\mathcal{L}\{f(t)\} = F(s)$, then
   
   (a) $\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$, and
   
   (b) $\mathcal{L}^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$.

2. Using the definition of the Laplace transform find
   
   $$\mathcal{L}\left[e^{at} \cos(at)\right].$$

3. From the Laplace transform of $\sin(at)$ and of $e^{-bt}$, find the Laplace transform of
   
   (a) $\cos(at)$
   
   (b) $\cos^2(at)$

4. Compute the inverse Laplace transform of
   
   $$\frac{5}{s^7} + \frac{7}{s-a}.$$

5. Compute the inverse Laplace transform of
   
   $$\frac{2s + 5}{s^2 - 25}.$$

6. Compute the inverse Laplace transform of
   
   $$\frac{2s + 14}{(s - 2)^2 + 36}.$$

7. Using the Laplace transform solve
   
   $$y'' + 9y = 12 \cos(3t)$$
   
   $$y(0) = 2 \quad y'(0) = 5$$
8. Using the Laplace transform solve
\[ y + \int_0^t y(\tau)d\tau = \sin(2t). \]

9. Using the Laplace transform solve
\[
\begin{align*}
y'' + 3ty' - 6y &= 1 \\
y(0) &= y'(0) = 0.
\end{align*}
\]

10. Using the Laplace transform solve and sketch the solution of
\[
\begin{align*}
x' - x &= e^{-t}H(t - 3) \\
x(0) &= 0.
\end{align*}
\]

11. Using the Laplace transform solve and sketch the solution of
\[
\begin{align*}
x'' + 2x' + x &= 10\delta(t - 5) \\
x(0) &= x'(0) = 0.
\end{align*}
\]

12. Let \( N(t) \) be the population of a bacterial colony at time \( t \). Let \( R(t) = \frac{dN}{dt} \), be the rate of change of the population. In the simplest model of population growth, \( R(t) = rN(t) \); so that the production rate is proportional to the total population. In a model which includes age structure in the population, the number of individuals born between times \( \tau \) and \( \tau + \delta \tau \) is \( R(\tau)\delta \tau + 0(\delta \tau)^2 \). At time \( t \), these individuals reproduce at growth rate \( \lambda(t - \tau) \) depending on their age \( t - \tau \). Hence
\[
R(t) = N_0\lambda(t) + \int_0^t \lambda(t - \tau)R(\tau)d\tau,
\]
where \( N_0 \) = initial population.

(a) Assuming \( \tilde{R}(s) \) exists, show
\[
\tilde{R} = \frac{N_0\lambda}{1 - \lambda}
\]

(b) Show that when \( \lambda = r \) (constant), \( R = N_0re^{rt} \) and \( N = N_0e^{rt} \)

(c) Find \( R(t) \) and \( N(t) \) when \( \lambda = e^{-kt} \) (productivity dies exponentially with age).