M&Ms Providing Insight on Packing Efficiencies

Abstract

Packing of spheres and other solids has been studied since at least 1611. For spheres, scientists have determined packing efficiencies, known as the packing fraction $\phi$, which is a ratio of the total volume occupied by the spheres to the total volume of the container. The packing efficiency for various methods of sphere packing, including structured and random packing has been considered.

Besides the random packing of spheres, the study of packing efficiencies of other solids has been studied. Specifically, the effect of increasing the magnitude of deviation from a spherical figure is of great interest. Being roughly the same shape and size, M&M’s Chocolate Candies provide a sufficient test object. When poured randomly and lightly shaken, M&M’s Chocolate Candies pack more densely than randomly packed spheres. After determining the volume of a single M&M and the total number of M&M’s in a container in comparison to the volume of the container, it is evident that M&M’s comprise a greater total volume of the container. M&M’s shape in particular, the oblate spheroid (a squashed spheroid in which the polar radius is greater than the equatorial radius), has a larger random packing efficiency ($\phi \approx .685$) than randomly packed spheres ($\phi \approx .64$).
The volume of a single M&M is .6205 ml. Using this information along with the packing efficiency $\varphi \approx 0.685$ for regular M&M’s, a hypothesis can be made regarding the number of M&Ms in a specific container of known volume. The container in this particular experiment had an estimated volume of 110 ml. Taking these factors into consideration, we estimate that there are 121 M&Ms in the container.

**Introduction**

The study of how particles pack was believed to be first considered by Johannes Kepler in 1611. He suggested that the densest possible arrangement of spheres fills 74.04% of the total space of a container. This remained unproven until 1998.

When discussing how particles pack, a specific ratio is considered. Represented by $\varphi$, the packing efficiency is defined to be the ratio of volume occupied by some solid to the total volume of the container in which the solids have been placed. Packing efficiencies vary as a function of the shape of a particular solid.

By first examining the extensive research and study concerning spherical packing, a comparison can be made to aspherical shapes. The packing fraction of spheres for various crystal lattice structures has been found through various experimentations. Specifically, a simple cubic lattice structure (Fig. 1) has a packing efficiency of 52%, while another known as body centered cubic (Fig. 2) has a packing efficiency of about 68%. The lattice structure with the most efficient packing fraction is the face-centered cubic (Fig. 3), in which $\varphi \approx 0.74$. As evidenced by the packing fractions, varying the lattice structure of the spheres greatly alters its packing efficiency. Another area of concern regards the packing efficiency of randomly packed spheres, as well as other similar surfaces. Through experimentation, scientists have found that randomly packed
spheres poured into a container and lightly shaken have a packing fraction approximately equal to .64, which is the arrangement with the largest random packing fraction for spheres and receives a special term known as random close packing. A similar concept, known as random lose packing, yields a packing fraction of .56, in which spheres are packed in a liquid in order to eliminate the effect of gravity. However, this packing technique represents the lowest packing fraction in which spheres are locked in place. Although these figures and techniques have theoretical importance, spheres are rarely the shape used in packing.

![Fig. 1. Simple Cubic](image1.png)  
**Fig. 1. Simple Cubic**  
Model of lattice structure with a low packing fraction ($\phi = .52$) [5].

![Fig. 2. Body Centered Cubic](image2.png)  
**Fig. 2. Body Centered Cubic**  
Model of lattice structure of average-high packing fraction ($\phi = .68$) [5].

![Fig. 3. Face Centered Cubic](image3.png)  
**Fig. 3. Face Centered Cubic**  
Model of lattice structure with most efficient packing fraction ($\phi = .74$) [5].

Recently, experiments have been performed to evaluate the random packing tendencies of spheroids, as well as ellipsoids. By deviating from a spherical shape, the packing efficiency for randomly packed objects is greatly altered. Through experimentation with M&M’s (oblate spheroids) and other shapes, scientists can begin to interpret the relationship between packing density and particle shape, as well as consider the impact of various methods of shaking and pouring. The exploration of packing efficiencies of various shapes reveals pertinent information in the field of mathematics in
regards to theoretical formulations and in the field of science in regards to particle behavior, which has many practical applications.

Knowing the packing fraction for M&Ms, we were able to predict the number of M&Ms in a specific container through several calculations. First we determined the volume of a single M&M by measuring the equatorial radius \(a\), which is approximately .67 cm and the polar radius \(b\), which is approximately .33 cm. From these measurements, the volume of an M&M can be expressed by the following equation:

\[ V = \frac{4}{3}a^2b = \frac{4}{3}(.67^2)(.33) = .6205 \text{ cm}^3 = .6205 \text{ ml}. \]

By multiplying the volume of the container (110 ml) by the packing efficiency \((\phi \approx .685)\) and then dividing by the volume of a single M&M (.6205 ml), the theoretical number of M&Ms can be determined:

\[ 110 \text{ ml}(.685)/(.6205 \text{ ml}) \approx 121. \]

By using M&Ms as a test subject, we sought to confirm the known packing efficiencies for M&Ms through several experiments and then apply our results to estimate an unknown number of M&Ms in a container of a specific volume. To increase the accuracy of our estimate, an approximation is also found by comparing the mass of the M&Ms in the container to the total mass of the container including the candy. Our analysis regarding packing efficiencies is then extended to consider packing efficiencies of other aspherical solids. In addition, the intersection of a cylinder and M&M is considered, as well as the path length of this intersection.

**Experimental**

Although numerous experiments have been recently performed using M&M Chocolate Candies that suggest that the packing efficiency of this oblate spheroid is
approximately $\varphi=0.685$, we sought to confirm this result by performing our own similar experiment.

A cylinder with diameter 6.5 centimeters was filled with M&Ms to certain level. Then by using the equation $\#\text{M&Ms}=\frac{V\times0.685}{0.6205\text{ml}}$, ($V=$volume occupied by the M&Ms including void space measured in ml.), we can check that the equation comes out with the correct number of M&Ms put into the jar. (M&Ms were counted before hand)

In the case of 110 M&Ms:

The M&Ms reached a height of 3 cm: $3\text{ cm}\times3.25\text{ cm}^2\pi = \text{volume occupied. This is approximately } 99.549\text{ cm}^3$. Plugging $V$ into our equation, we end up with 109.9 as an answer, which is nearly the 110 M&Ms put in the jar.

Fig. 4. Representation of M&M packing [6].

In the case of 165 M&Ms:

The M&Ms reached a height of 4.5 cm: $4.5\text{ cm}\times3.25\text{ cm}^2\pi =149.324\text{ cm}^3$. Plugging $V$ into our equation, we end up with 164.846 which is nearly the 165 M&Ms put in the jar.

In the case of 220 M&Ms:

The M&Ms reached a height of 6 cm: $6\text{ cm}\times3.25\text{ cm}^2\pi =199.098\text{ cm}^3$. Plugging $V$ into our equation, we end up with 219.794 which is nearly the 220 M&Ms put in the jar.
(Note: To confirm our original estimate of the number of M&Ms in the jar, another simulation was used to determine the theoretical number of M&Ms based on mass. By first finding the mass of the jar filled with M&Ms (331.9 grams) and then measuring the mass of the empty jar (236.88 grams), we were able to determine the total mass of the M&Ms (331.9-236.88 = 95.02 grams). This amount was then divided by the average mass of a single M&M (.82 grams). Based on these measurements, the theoretical number of M&Ms in the jar is approximately (95.02 grams/.82 grams) 116, which is close to our first and primary estimate of 121 M&Ms.)

<table>
<thead>
<tr>
<th>Method of Calculation</th>
<th>M&amp;M Prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume and Packing Fraction</td>
<td>121</td>
</tr>
<tr>
<td>Mass</td>
<td>116</td>
</tr>
</tbody>
</table>

**Table 1. Table comparing the results of the two methods used to determine the number of M&Ms in the container.**

**Theory**

The density achieved by random packing is dependent on the average number of items that come in contact with any one item in the packing [2]. The relation is shown in the following graphs:
The graph includes data for both prolate (○) and oblate (□) ellipsoids as well as fully aspherical (◊) ellipsoids. Figure 5 shows that the maximum volume fraction is achieved when the items deviate from a sphere with its aspect ratio of about 1.25 where the aspect ratio $\alpha = b/a$, for ellipsoids: $x^2/a + y^2/b + z^2/c = 1$. For this aspect ratio, the average coordination is about 11.3. This means that for the greatest volume fraction, the average number of contact points is about 11. An M&M has an aspect ratio of 0.5: $b/a = .33/.67$. This aspect ratio yields a volume fraction between 0.68 and 0.70. This ratio indicates that M&Ms randomly fill between 68% and 70% of their container.

Jellybeans are in the shape of an aspherical ellipsoid. Using the graph of volume fraction vs. aspect ratio and assuming the aspect ratio of a jellybean is between 1.3 and 2.2, the random packing should yield a volume fraction between 0.7 and 0.715. These volume fractions are greater than the volume fraction of an M&M. These fractions imply
that the more a figure deviates from a sphere to a specific point, the greater the packing efficiency.

The increased packing efficiency associated with aspherical shapes results from the increased number of contact points between the particles. Because of its shape, ellipsoids have more degrees of freedom, or an increased number of directions in which the particle can move. Each surrounding object that touches the central particle applies a force on that particle. In order for that particle to be in equilibrium, these forces in each direction must sum to be zero. In the case of spheres, exerted forces can cause the sphere to move but not rotate, whereas in the case of ellipsoids, exerted forces can cause it to move and rotate. In order to stabilize the particles, more contact points are required in order to eliminate the increased number of degrees of freedom. The increased amount of contact points results in a more compact packing and larger packing efficiency.

More specifically, the exact volume fraction of a jellybean or other candy can be found by first determining the values for $a$ and $b$, where $a =$ length of the radius on the horizontal axis, while $b =$ length of the radius on the horizontal axis. The aspect ratio ($\alpha$) for any solid can then be computed according to the following: $\alpha = b/a$. This value of $\alpha$ can then be used to determine the volume fraction or packing efficiency for that particular solid by locating the aspect ratio on Figure 5.
Another area of interest results from the intersection of a cylinder with a sphere (Figure 8). This intersection is between a cylinder of radius \( a \), center \( 2a \) and of the general equation: \((x-a)^2 + y^2 = a^2\) and the sphere of radius \( 2a \), center \((0,0,0)\) and of the general equation: \(x^2+y^2+z^2=(2a)^2\). Known as Viviani’s Curve, it can be applied in looser terms to the intersection of a cylinder with an M&M (oblate spheroid) where the radius \((a)\) of the cylinder is found by dividing the equatorial radius or the longer of the radii by two. In the case of an M&M, \( a = .335 \) cm

\[
\begin{align*}
\text{M&M:} & \quad \frac{x^2 + y^2}{.67^2} + \frac{z^2}{.33^2} = 1 \\
\text{Cylinder:} & \quad (x-.335)^2 + y^2 = .335^2
\end{align*}
\]

![Fig. 8. Representation of M&M (red) with intersecting cylinder (yellow).](image-url)
The intersection of these two surfaces (Figure 9) can be defined by the following parametric equations:

\[ x = .335 \left( \cos(t) + 1 \right) \]
\[ y = .335 \sin(t) \]
\[ z = \sqrt{1 - \left( \frac{.335 \left( \cos(t) + 1 \right)}{.67} \right)^2 - \left( \frac{.335 \sin(t)}{.67} \right)^2} \cdot .33^2 \]

By putting the equation for this space curve in the form of a vector, the path length of the intersection can be determined:

\[ \mathbf{r}(t) = \langle .335 \left( \cos(t) + 1 \right), .335 \sin(t) \rangle \sqrt{1 - \left( \frac{.335 \left( \cos(t) + 1 \right)}{.67} \right)^2 - \left( \frac{.335 \sin(t)}{.67} \right)^2} \cdot .33^2 \]

Path Length\( = 2 \cdot \int_{-\pi}^{\pi} |\mathbf{r}'(t)| \, dt \approx 4.454 \text{ cm} \)

**Fig. 9. Representation of the curve of intersection between the cylinder and M&M.**
Discussion

We were able to analyze the packing efficiencies of M&Ms and apply the results to similar aspherical objects. By measuring the volume of a single M&M and determining the packing efficiency for M&Ms, we were able to make a conjecture about the number of M&Ms in a container of a specific volume. This knowledge can also be applied to a container of a different shape or volume. Information concerning the packing efficiency of M&Ms is not only useful in predicting the amount of candy in jar but can be applied in a larger sense to other similar shapes with various practical uses.

The larger packing fraction for randomly packed oblate spheroids and other ellipsoids in comparison to spheres has several important applications. One specific area of interest is the implication for packing and shipping. Research suggests that particles are able to be packed more compactly the greater their shape deviates from a sphere. This information provides insight into methods of packing particles in a smaller volume.

Another field impacted by this discovery is in material science concerning ceramics. Ceramics are made by combining miniscule particles of powder. By choosing particle shapes that share numerous contact points, scientists can form stronger, less permeable ceramics.

In addition, the strength of various types of glass can further be analyzed in that glass consists of randomly arranged liquid particles that have become stuck in place. By studying the shape of the molecules, scientists can better understand the formation and properties similar to packing efficiencies of glass. These ideas can also offer new information to similar substances that have combined properties of solids and liquids.

The random arrangement of M&Ms provides great insight in many fields of mathematics.
and science.
Works Cited


