Homework Set 8 Solutions

1. Consider the following equation:
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \quad u(x,0) = x - 2n, \quad x \in (2n - 1, 2n + 1), \quad n \in \mathbb{Z}. \quad (8.1) \]

(a) (10 points) Construct the solution. Show that the shock positions are stationary, and the shock strength decays like \( t^{-1} \) as \( t \to \infty \).

Solution. Solving, we obtain
\[
\frac{du}{dt} = 0 \quad \text{when} \quad \frac{dx}{dt} = u
\]
\[ u = u(\xi, 0) \quad \text{when} \quad x = u(\xi, 0)t + \xi
\]
\[ x = (\xi - 2n)t + \xi, \quad \xi \in (2n - 1, 2n + 1),
\]
\[ \xi = \frac{x + 2nt}{1 + t}
\]
\[ u(x,t) = u(\xi,0) = \xi - 2n = \frac{x + 2nt}{1 + t} - 2n
\]
\[ = \frac{x - 2n}{1 + t}, \quad x \in (2n - 1, 2n + 1). \]

This solution is discontinuous at \( x = 2n + 1, \ t = 0 \), so we must introduce a shock \( x = s(t) \), keeping in mind that \( q = u^2/2 \):
\[
\frac{ds_n}{dt}[u(s_n^+,t) - u(s_n^-,t)] = q(s_n^+,t) - q(s_n^-,t)
\]
\[
\frac{ds_n}{dt} = \frac{u^2(s_n^+,t) - u^2(s_n^-,t)}{2[u(s_n^+,t) - u(s_n^-,t)]} = \frac{u(s_n^+,t) + u(s_n^-,t)}{2}
\]
\[ = \frac{1}{2} \left\{ \frac{s_n - 2(n + 1)}{1 + t} + \frac{s_n - 2n}{1 + t} \right\}
\]
\[ = \frac{s_n - 1 - 2n}{1 + t}
\]
\[ \log(s_n - 1 - 2n) = \log(1 + t) + \log A
\]
\[ s_o(t) = A(1 + t) + 2n + 1 = 2n + 1.
\]

Hence the shock is stationary. The strength of the shock is given by
\[
u(s_n^+,t) - u(s_n^-,t) = \left\{ \frac{2n + 1 - 2(n + 1)}{1 + t} - \frac{2n + 1 - 2n}{1 + t} \right\}
\]
\[ = -\frac{2}{1 + t}. \]
which decays like $t^{-1}$ as $t \to \infty$. (Alternatively, one can also take the absolute value of the above.)

(b) (4 points) Draw the characteristic diagram, indicating the position of the shocks.

**Solution.**

![Characteristic diagram for #1. This diagram repeats with period 2. The shock at $2n + 1$ is indicated with a thick line.]

(c) (4 points) Sketch $u$ for various values of $t$.

**Solution.** See next page.

2. The telegraph equation has the form

$$\frac{\partial^2 u}{\partial x^2} = LC \frac{\partial^2 u}{\partial t^2} + (RC + GL) \frac{\partial u}{\partial t} + RGu,$$

where all the parameters are positive.

(a) (5 points) Let $u = v(x,t)e^{\alpha t}$ and choose $\alpha$ such that the differential equation for $v$ has no $\partial v/\partial t$ term.

**Solution.** Introducing this substitution into (8.2), we obtain

$$\frac{\partial^2 v}{\partial x^2} e^{\alpha t} = LC \left( e^{\alpha t} \frac{\partial^2 v}{\partial t^2} + 2\alpha e^{\alpha t} \frac{\partial v}{\partial t} + \alpha^2 e^{\alpha t} v \right) + (RC + GL) \left( e^{\alpha t} \frac{\partial v}{\partial t} + \alpha e^{\alpha t} v \right) + RG e^{\alpha t} v$$

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2} + (2\alpha LC + RC + GL) \frac{\partial v}{\partial t} + [\alpha (\alpha LC + RC + GL) + RG] v. \quad (A)$$
Profiles of $u$ for $t = 0, 1, 2$ (in increasing order of thickness).

Therefore, by setting

$$\alpha = -\frac{RC + GL}{2LC},$$

we may zero out the first-derivative term in (A).

(b) (3 points) Show (by calculation, **NOT** direct substitution) that if $RC = GL$, your answer to (a) reduces to the standard wave equation for $v$.

Solution. Using (A), we must have that

$$\alpha(\alpha LC + RC + GL) + RG = 0$$

$$\left(-\frac{RC + GL}{2LC}\right)\left(\frac{RC + GL}{2}\right) + RG = 0$$

$$(RC + GL)^2 - 4RCGL = 0$$

$$(RC - GL)^2 = 0$$

$$RC = GL.$$

With this substitution, (A) becomes

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 v}{\partial t^2}, \quad c^2 = (LC)^{-1}.$$  \hspace{1cm} (B)

(c) (4 points) Show that in the case described by (b), a signal arriving at $x = b$ is simply a damped version of a signal sent at $x = 0$. 
Solution. The general solution for (B) is given by

\[ v(x,t) = f(x - ct) + g(x + ct) \]
\[ u(x,t) = e^{\alpha t} [f(x - ct) + g(x + ct)]. \]

Any signal sent at \( t = 0 \) will have the form \( f(x) + g(x) \). The above shows that the profile will not change, it will just decay by an amount \( e^{\alpha t} \) (recall that \( \alpha < 0 \)).

3. Consider the flow and density given by

\[ x = a, \quad y = b + c \cos \omega t, \quad \rho = \frac{\rho_0}{1 + y^2}. \quad (8.3) \]

(a) (2 points) Show that the flow in (8.3) is not steady. Describe it.

Solution. The velocity field is given by

\[ v_x = 0, \quad v_y = -c\omega \sin \omega t. \]

Since \( v_y \) depends on \( t \), the flow is not steady. Rather it is a flow which is oscillating with amplitude \( c \) and frequency \( \omega \) in the \( y \)-direction only.

(b) (3 points) Calculate \( \frac{D\rho}{Dt} \).

Solution. We have

\[ \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} \]
\[ = 0 + 0 - c\omega \sin \omega t \left[ -\frac{2y\rho_0}{(1 + y^2)^2} \right] \]
\[ = \frac{2yc\omega \rho_0 \sin \omega t}{(1 + y^2)^2}. \]

4. (5 points) Suppose that a fluid is in steady motion past a bounded obstacle, and further suppose that \( f \equiv 0 \). Let \( \mathbf{K} \) be the force acting on the obstacle. Use the linear momentum transfer equation:

\[ \frac{d}{dt} \int_R \rho \mathbf{v} \, dV = \int_R \rho \mathbf{f} \, dV + \int_{\partial R} [\mathbf{t} - \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n})] \, dA, \quad (8.4) \]

to deduce that

\[ \mathbf{K} = \int_S [\mathbf{t} - \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n})] \, dA, \quad (8.5) \]

where \( S \) is any (imaginary) surface enclosing the obstacle. (In practice \( S \) is chosen to simplify calculations). Be sure to explain why the sign of \( \mathbf{K} \) is as indicated.
Solution. Let $R$ be a hollow region whose inner boundary $O$ is given by the obstacle and whose outer boundary is given by $S$. Then with $f \equiv 0$ in (8.4), we have

$$\int_{R} \frac{\partial(\rho \mathbf{v})}{\partial t} \, dV = \int_{\partial R} [t - \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n})] \, dA$$

$$0 = \int_{O} [t - \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n})] \, dA + \int_{S} [t - \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n})] \, dA$$

$$- \int_{O} t \, dA = \int_{S} [t - \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n})] \, dA. \quad (C)$$

On the first line we used the fact that since the region isn’t moving, we may bring the time derivative inside the integral. On the second line we used the fact that the flow is steady, and on the third line we used the fact that on the obstacle the normal velocity must be zero. The left-hand side of (C) is the negative of the contact force which the obstacle exerts on the fluid. But by Newton’s third law this is exactly the force which the fluid exerts on the obstacle, denoted by $K$. Therefore, we have

$$K = \int_{S} [t - \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n})] \, dA,$$

as required.