Phase Plane Portraits

For the system
\[
\begin{pmatrix}
3 & 4 \\
4 & -3
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix},
\]
\[\dot{x} = \begin{pmatrix}
3 & 4 \\
4 & -3
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix}, \tag{1}
\]
the solution is
\[
x = c_1 e^{5t} \begin{pmatrix}
2 \\
1
\end{pmatrix} + c_2 e^{-5t} \begin{pmatrix}
1 \\
-2
\end{pmatrix}.
\]

Since we have one positive and one negative eigenvalue, we have a saddle point, as shown below. Note the straight lines corresponding to the eigenvectors.
For the system
\[
\dot{x} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} x,
\]
the solution is
\[
x = c_1 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]
Since we have two negative eigenvalues, we have a stable node, as shown below.

Phase plane of (2).
For the system
\[
\dot{x} = \begin{pmatrix} 3 & -5 \\ 5 & -3 \end{pmatrix} x,
\] (3)
the solution is
\[
x = c_1 \begin{pmatrix} 5 \cos 4t \\ 3 \cos 4t + 4 \sin 4t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin 4t \\ 3 \sin 4t - 4 \cos 4t \end{pmatrix}.
\]
Since there are no real eigenvectors, all trajectories spin about the origin. Since the real part of the eigenvalues is zero, the origin is a center, as shown below:

Phase plane of (3).
For the system
\[
\dot{x} = \begin{pmatrix} -2 & -6 \\ 3 & 4 \end{pmatrix} x,
\]  \hspace{1cm} (4)
the solution is
\[
x = c_1 e^{t \begin{pmatrix} -\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix}} + c_2 e^{t \begin{pmatrix} \cos 3t - \sin 3t \\ \sin 3t \end{pmatrix}}.
\]
Since the real part of the eigenvalues is positive, we have an unstable spiral, as shown below.

![Phase plane of (4).](image-url)