Math 426 Intro to Numerical Analysis and Algorithmic Computing 03F

R. J. Braun

Assignment 5, due Monday 11/10/03, 4:30pm

1. Ch. 7, §1: 1(b,d),3(a),4(a,b),5(a),6(a,b)
2. Ch. 7, §2: 1(a,c),2(a,c),3(a,c),4
3. Ch. 7, §3: 1(b),2(b),5(b),12
4. Ch. 7, §4: 1(d),2(d)
5. Ch. 7, §6: Read.
6. Note that you may use Matlab to do the calculations above.

Solutions, Answers and Hints

Section 7.1

1(b). \( x = (2, 1, -3, 4)^T \). \( ||x||_\infty = \max_i |x_i| = 4 \). \( ||x||_2 = \sqrt{\sum_i x_i^2}^{1/2} = \sqrt{16} = 4 \). \( ||x||_\infty = \max_i |x_i| = 4 \).

(d). \( x = (4/(k+1), 2/k^2, k^2e^{-k})^T \). We have

\[ k = 1 : \quad |x_1| = |x_2| = 2 > |x_3| = e^{-1}; \]

\[ k = 2 : \quad |x_1| = 4/3 > |x_3| = 4e^{-2} > |x_2| = 1/2; \]

so \( ||x||_\infty = |x_1| = 4/(k+1) \) for this problem. For the \( \ell_2 \)-norm, we have

\[ ||x||_2 = \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2} = \sqrt{\frac{16}{(k+1)^2} + \frac{4}{k^2} + k^4e^{-2k}}. \]

3(a). We have to show that each element has a limit, and find it. Each individul has a limit of 0, so the limit exists and it is \((0, 0, 0)^T\).

4(a). \( ||A||_\infty = \max \{ |10| + |15|, |0| + |1| \} = 25. \)

4(b). \( ||A||_\infty = \max \{ |10| + |0|, |15| + |1| \} = 16. \)

5(a). \( ||\tilde{x} - x||_\infty = \max \{ |0.142 - 1/7|, | - 0.166 - (-1/6)| \} = 8.57 \times 10^{-4} \) (from the first of the two). \( A\tilde{x} - b = (-2.06349 \times 10^{-4}, -1.19048 \times 10^{-4})^T \); then \( ||A\tilde{x} - b||_\infty = 2.06349 \times 10^{-4}. \)

6(a). \( ||A||_1 = \max \{ |10| + |0|, |15| + |1| \} = 16. \)

6(b). \( ||A||_1 = \max \{ |10| + |15|, |0| + |1| \} = 25. \)
Section 7.2

1(a). \( \det(A - \lambda I) = 0 \) gives \((2 - \lambda)^2 - 1 = 0 \) or \( \lambda - 2 = \pm 1 \); then \( \lambda = 1, 3 \). With \( \lambda = 1 \), \( Ax = x \) and one finds the eigenvector \( x = (1, 1)^T \). With \( \lambda = 3 \), \( Ax = 3x \) and one finds the eigenvector \( x = (1, -1)^T \).

1(c). \( \det(A - \lambda I) = 0 \) gives \( (\lambda^2 - 1/4) = 0 \) or \( \lambda = \pm 1/2 \). With \( \lambda = -1/2 \), \( Ax = -x/2 \) and one finds the eigenvector \( x = (1, -1)^T \). With \( \lambda = 1/2 \), \( Ax = x/2 \) and one finds the eigenvector \( x = (1, 1)^T \).

2(a). \( \rho(A) = \max_i |\lambda_i| = 3 \).

2(c). \( \rho(A) = \max_i |\lambda_i| = 1/2 \).

3(a). Since \( \rho(A) > 1 \), \( A \) is not convergent.

3(c). Since \( \rho(A) < 1 \), \( A \) is convergent.

4. Here \( A_1 = [1 \ 0; 1/4 \ 1/2] \), we have

\[
A_1 A_1 = A_1^2 = \begin{bmatrix} 1 & 0 \\ 3/8 & 1/4 \end{bmatrix} ; \quad A_1 A_1^2 = A_1^3 = \begin{bmatrix} 1 & 0 \\ 7/16 & 1/8 \end{bmatrix} ;
\]

one finds that

\[
A_1^k = \begin{bmatrix} 1 \frac{2^k+1}{2^{k+1}} \\ \frac{1}{2^k} \end{bmatrix} \quad \text{and} \quad \lim_{k \to \infty} A_1^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Therefore \( A_1 \) is not convergent.

Now for \( A_2 = [1/2 \ 0; 16 \ 1/2] \),

\[
A_2 A_2 = A_2^2 = \begin{bmatrix} 1 & 0 \\ 16/4 & 1/4 \end{bmatrix} ; \quad A_2 A_2^2 = A_2^3 = \begin{bmatrix} 1/8 & 0 \\ 12 & 1/8 \end{bmatrix} ;
\]

one finds that

\[
A_2^k = \begin{bmatrix} 2^{-k} & \frac{2^k}{16k-1} \\ 0 & 2^{-k} \end{bmatrix} \quad \text{and} \quad \lim_{k \to \infty} A_2^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Therefore, \( A_2 \) is convergent.

Section 7.3

1(b), 2(b), 5(b). Answers for each of these are in the following diary included here.

% Assignment 5, Ch 7 iterative methods.
% Problem 7.3.1(b) (Gauss-Jacobi)
>> GJCh7
A =
10 -1 0
-1 10 -2
0 -2 10
b =
9
% Problem 7.3.2(b) (Gauss-Seidel)
>> GSCh7
A =
10  -1   0
-1   10  -2
 0  -2   10 
b =
 9 
 7 
 6 
GS step? (0=no, 1=yes)1
k =
 1 
x =
 0.9000 
 0.7000 
 0.6000
GS step? (0=no, 1=yes)1
k =
 2 
x =
 0.9700 
 0.9100 
 0.7400
GS step? (0=no, 1=yes)0
% Problem 7.3.5(b) (SOR with omega=1.1)
>> SORCh7
A =
10  -1   0
-1   10  -2
 0  -2   10 
$$b =  
\begin{align*}
9 
7 
6 
\end{align*}$$

SOR step? (0=no, 1=yes)1

$$k = 1$$

$$x =  
\begin{align*}
0.9900 
0.8789 
0.8534 
\end{align*}$$

SOR step? (0=no, 1=yes)1

$$k = 2$$

$$x =  
\begin{align*}
0.9877 
0.9785 
0.7899 
\end{align*}$$

SOR step? (0=no, 1=yes)0

>> diary off

12. For Gauss-Jacobi, $T_j = t_{ij}$ with $t_{ii} = 0$ and $t_{ij} = -a_{ij}/a_{ii}$ with $i \neq j$. By the definition of the $\ell_\infty$-norm,

$$||T_j||_\infty = \max_i \sum_{j=1}^n |t_{ij}| = \frac{1}{|a_{ii}|} \sum_{j=1}^n |a_{ij}|;$$

by the definition of a diagonally dominant matrix $A = a_{ij}$, this sum for any $i$ is less than one. Therefore, $||T_j||_\infty < 1$.

Section 7.4

1(d). $||A||_\infty = 327.35$; also,

$$A^{-1} = \begin{bmatrix} 879.4753 & -158.7592 \\ -15.1681 & 2.7412 \end{bmatrix}$$

and so $||A^{-1}||_\infty = 1038.2$. Then $K_\infty(A) = ||A||_\infty||A^{-1}||_\infty = 3.399 \times 10^5$.

2(d). $||x - \bar{x}||_\infty = 20$; $b - A\bar{x} = (20.06, 111.0)^T$ and so $||b - A\bar{x}||_\infty = 111.0$. Then, using the result from 7.4.1(d),

$$K_\infty(A) \frac{||b - A\bar{x}||_\infty}{||A||_\infty} = 115,200$$

to four figures.