Math 426 Intro to Numerical Analysis and Algorithmic Computing 03F
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Assignment 1

1. Ch. 1, §1: 1(b), 3(c), 9, 14.
2. Ch. 1, §2: 1(a), 2(b), 5(c,d), 15(b), 16(b), 25.
3. Ch. 1, §3: 2(a,b), 6(a-c), 7(a).
4. Read §1.4.
5. Begin by using \(E_{20} = 0\) and compute the six digit (decimal) result for \(E_9\) using
   \[E_n = \frac{1}{n!} \int_0^1 x^n e^{-1} \, dx.\]

Solutions

Chapter 1, Section 1

1(b). Let \(f(x) = (x-2)^2 - \ln x\). For the interval \(x \in [1, 2]\) we have \(f(x) \in C[1, 2]\) with \(f(1) = 1 > 0\) and \(f(2) = -\ln 2 < 0\); by the Intermediate Value Theorem, there exists \(p \in [1, 2]\) where \(f(p) = 0\). For the interval \(x \in [e, 4]\) we have \(f(x) \in C[e, 4]\) with \(f(e) = (e - 2)^2 - 1 < 0\) and \(f(4) = 4 - \ln 4 > 0\); by the Intermediate Value Theorem, there exists \(p \in [e, 4]\) where \(f(p) = 0\).

3(c). \(f(x) = 1 - e^x + (e - 1) \sin(\pi x/2)\) is continuous and differentiable on \([0, 1]\), with \(f(0) = 0\) and \(f(1) = 0\). By Rolle’s Theorem, there is at least one \(c \in (0, 1)\) such that \(f'(c) = 0\).

9. The second Taylor polynomial of \(f(x)\) about \(x_0\) is given by
   \[P_2(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2;\]
   for our problem, with \(x_0 = 0\) and \(f(x) = e^x \cos x\), we have
   \[P_2(x) = 1 + \frac{1}{1!}(x - 0) + \frac{0}{2!}(x - 0)^2 = 1 + x.\]
   The remainder term is
   \[R_2(x) = \frac{f'''(\xi(x))}{3!}(x - x_0)^3 = \frac{-2e^\xi(x)(\sin \xi(x) + \cos \xi(x))}{3!}(x - 0)^3.\]
(a) \( P_2(0.5) = 1.5 \); The absolute error can be bounded as follows:

\[
|f(1.5) - P_2(0.5)| = |R_2(0.5)| = \left| \frac{f'''(\xi(0.5))}{3!} (0.5)^3 \right| \\
= \left| \frac{-2e^{\xi(0.5)}(\sin(0.5) + \cos(0.5))}{3!} (0.5)^3 \right| \\
\leq \left[ \max_{x \in [0,0.5]} |f'''(x)| \right] \left| \frac{(0.5 - 0)^3}{3!} \right|.
\]

Now from analyzing \( f''' \) using calculus or graphing \( f''' \) in MATLAB, \( |f'''(\xi(0.5))| \leq |f'''(0.5)| \approx 4.475 \); substitution into \( R_2 \) gives that the absolute error is bounded by \( |R_2(x)| \leq 0.0932 \). The actual error is \( |f(0.5) - P_2(0.5)| = 1.447 - 1.5 = 0.053 \), which is less than the error bound.

(b) Now find an error bound for \( |f(x) - P_2(x)| \) on \( x \in [0, 1] \). Now

\[
|R_2(x)| \leq \left[ \max_{x \in [0.0, 1]} |f'''(x)| \right] \left| \frac{1 - 0)^3}{3!} \right| = |f'''(1)| \frac{1^3}{3!} = 7.512 \frac{3!}{3!} = 1.252.
\]

(c) For the integral, we have \( \int_0^1 f(x) \, dx \approx \int_0^1 (1 + x) \, dx = 1.5 \).

(d) To bound the error, bound \( R_2(x) \) inside the integral by bounding \( f'''(x) \) first, then integrate:

\[
\int_0^1 |R_2(x)| \leq \int_0^1 \left[ \max_{x \in [0,1]} |f'''(x)| \right] \left| \frac{x^3}{3!} \right| \, dx = \frac{7.512}{4!} \left| x^4 \right|_0^1 = 0.313.
\]

Thus the bound on the error is 0.313; the actual error of 0.122 is smaller.

14. Here the remainder term from expanding about \( x_0 = 0 \) is \( R_2(x) = f'''(\xi(x))x^3/3! \). Here \( f'''(\xi(x)) = -\cos(\xi(x)) \), with \( \xi \) between 0 and 1 is bounded by \( f'''(0) = 1 \); using \( x = 1(\pi)/180 \) (radians), we have \( R_2(x) \leq 1 \times (\pi/180)^3/6 = 8.861 \times 10^{-7} \).

Chapter 1, Section 2

1(a). \( p^* = 22/7 \) and \( p = \pi \). Absolute error: \( |22/7 - \pi| = 0.001264489 \). Relative error: \( |22/7 - \pi|/|\pi| = 0.000402499 \).

2(b). \( \frac{|e - p^*|}{|e|} \leq 10^{-4} \); then

\[-10^{-4}e \leq e - p^* \leq 10^{-4}e.\]

Taking each side one at a time,

\[p^* \leq e(1 + 10^{-4}) = 2.7185537\]

\[p^* \geq e(1 - 10^{-4}) = 2.7180100.\]

Thus \( 2.7180100 \leq p^* \leq 2.7185537 \).

5(c). With 3-digit rounding, \( (121 - 0.327) - 119 = 121 - 119 = 2.00 \); the exact answer is 1.673. The absolute error is 0.327; the relative error is 0.195457.
5(d). With 3-digit rounding, \((121 - 119) - 0.327) = 2 - 0.327 = 1.67;\) the exact answer is 1.673. The absolute error is 0.003; the relative error is 0.00179319.

15(b). \(s = 1; c = 1 \times 2^{10} + 1 \times 2^3 + 1 \times 2^1 = 1034; f = 2^{-1} + 2^{-4} + 2^{-7} + 2^{-8} = 0.57421875.\) Then \(x = (-1)^{s}2^{c-1023}(1 + f) = -3224.\) Matlab displays this number with no decimal part (as an integer).

16(b). The next largest number is from decreasing the mantissa by 1 in the last place which gives

\[
1 10000001010 100100110000000000000000000000000000000000000000001
\]

Evaluating in a similar way to before gives \(-3224 - 2^{-41},\) or \(-3224 + 4.5474735 \times 10^{-13}.\) The next smallest number turns the very last digit in 15(b) into a 1, giving

\[
1 10000001010 100100100000000000000000000000000000000000000000001
\]

25. (a) Evaluating the definition as given requires computing, from left to right, \(m!/k!/(m - k)!;\) computing \(m!\) will cause the largest number during the calculation. Using, say, \texttt{factorial(m)} in \texttt{MATLAB}, \(m = 17\) has \(17! \approx 3.55 \times 10^{14},\) while \(18! \approx 6.4 \times 10^{15};\) therefore \(m = 17\) is the biggest integer that could be used in the given number system with this definition.

(b) Using the definition,

\[
\begin{align*}
\binom{m}{k} &= \frac{m(m - 1)(m - 2) \cdots (m - k + 1)}{[k(k - 1)(m - k) \cdots (2)1][(m - k)(m - k - 1) \cdots (2)1]} \\
&= \frac{m(m - 1)(m - 2) \cdots (m - k + 1)}{k(k - 1) \cdots (2)1} \\
&= \left( \frac{m}{k} \right) \left( \frac{m - k}{k - 1} \right) \cdots \left( \frac{m - k + 1}{1} \right)
\end{align*}
\]

(c) Using part (b), we are limited by the final computed number, which is 181707. One could estimate the number by noting that \(m^3/3 \approx 0.9999 \times 10^{15}\) when \(m\) is large (neglecting the lower degree terms from the equation of part (b)), and then trying numbers near the resulting \(m.\) One could also solve for \(m\) from the exact cubic equation from part(b). Note that if only four digits are allowed, then the largest allowed number would only be 181700, and that the next number, 181800, would cause overflow for this number system.
(d) With $m = 52$, $k = 5$, and 4 digit chopped arithmetic,

\[
\begin{bmatrix}
  52 \\
  5
\end{bmatrix}
= \begin{bmatrix}
  \frac{52}{5} \\
  \frac{51}{4} \\
  \frac{50}{3} \\
  \frac{49}{2} \\
  48
\end{bmatrix}
= (10.4)(12.75)(16.66)(24.5)(48)
\]

The exact answer is 2598960. Actual error is 1960; Relative error is $7.541 \times 10^{-4}$.

Chapter 1, Section 3

2(a). Using 4 digit chopping and summing from largest to smallest as specified, we have

\[
1 + 1 + 0.1666 + 0.04166 + 0.008333 = 2.666 + 0.04166 + 0.008333 = 2.707 + 0.008333 = 2.715.
\]

Absolute error is $|e - 2.715| = 0.00328$; relative error is $|2.715 - e|/e = 1.21 \times 10^{-3}$.

2(b). Using 4 digit chopping and summing from smallest to largest as specified, we have

\[
0.008333 + 0.04166 + 0.3333 + 0.5 + 1 = 0.04999 + 0.1666 + 0.5 + 1 + 1 = 2.716.
\]

Absolute error is $|e - 2.716| = 0.00228$; relative error is $|2.716 - e|/e = 8.39 \times 10^{-4}$.

6(a). Use Taylor’s theorem, thinking of $u = 1/n$ as small and expanding about $u = 0$ (or “$n = \infty$”); then

\[
\sin u \approx \cos(\xi_u(u))(u - 0) \leq u = \frac{1}{n}.
\]

Now the comparison sequence is $\beta_n = 1/n$ and the rate of convergence is $O(1/n)$.

6(b). Let’s try a different approach this time; use a Taylor series now.

\[
\sin \left( \frac{1}{n^2} \right) = \frac{1}{n^2} - \frac{1}{3!n^6} + \frac{1}{5!n^{10}} - \ldots
\]

\[
= \frac{1}{n^2} \left( 1 - \frac{1}{3!n^4} + \frac{1}{5!n^8} - \ldots \right)
\]

\[
< \frac{1}{n^2} \left( 1 + \frac{1}{n^4} + \frac{1}{n^8} + \ldots \right)
\]

\[
= \frac{1}{n^2} \left( \frac{1}{1 - \frac{1}{n^4}} \right)
\]

\[
< \frac{1}{n^2} \left( \frac{1}{1 - \frac{1}{2}} \right)
\]

\[
= \frac{2}{n^2}.
\]

Recall that we are considering a large $n$ limit for rate of convergence; now we have $K = 2$ and $\beta_n = 1/n^2$ so that the rate of convergence is $O(1/n^2)$. 

6(c). One could use either of the above approaches to arrive at an answer. Using the first approach, one obtains

\[ [\sin(u)]^2 = 2[\cos^2(\xi_u(u)) - \sin^2(\xi_u(u))](u - 0)^2/2! \leq 1/n^2. \]

Here \( K = 1 \) and \( \beta_n = 1/n^2 \), thus the rate of convergence is \( O(1/n^2) \).

7(a). Using Taylor’s theorem,

\[ 1 - \frac{\sin h}{h} = \cos(\xi(h))h^2/3! \leq h^2; \]

Thus the rate of convergence is \( O(h^2) \).

**Evaluating \( E_n \)**

(This problem is due to Forsythe, Malcolm and Moler.) Begin with \( E_{20} = 0 \) and use \( E_{n-1} = (1 - E_n)/n \); see the table. The exact answer to 13 digits is 0.09161229300662; not bad! Note that I didn’t stick to 6 digits here; sticking to six digits gets the first six digits right in \( E_9 \).

<table>
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<th>( n )</th>
<th>( E_n )</th>
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<td>0.08387707010</td>
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<tr>
<td>9</td>
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Table 1: Computing \( E_n \).