ABSTRACT: We present a finite element formulation of Achi Brandt’s “distributive relaxation” for the Stokes equations. The transformed system we get is almost block-lower triangular, and the principal diagonals are all Laplace-like matrices. The distributive relaxation is used as a smoother for the multigrid method. The effectiveness of this smoother is shown by numerical results.

1 Introduction
In stead of solving (smoothing) the original Stokes equations
\[
(-\nabla \cdot \mu \partial \mu = 0)
\]
We make the following transformation
\[
(\mu \partial \mu = 0)
\]
Then, we are going to smooth
\[
(-\nabla \cdot \mu \nabla \mu = 0)
\]
Notice that \(\nabla \cdot \nabla \mu = 0\) is zero inside the region, so we only need to smoother two Laplace problems.

2 Finite Element Method Case
For most stable finite element pairs, the discrete \(\nabla \nabla \mu = 0\) is not inside the region. Distributive relaxation seems not applicable. We choose the continuous piecewise quadratic functions space \(S_h\) on mesh \(T_h\) to approximate the pressure, and \(\mathbf{V}_h\) be the space of continuous piecewise linear functions w.r.t. the mesh \(T_h\). Let \(\mathbf{v}_h \in \mathbf{V}_h\) and \(Q_2\) is a quasi-interpolant to transfer the derivatives of \(\mathbf{v}_h\) into the space \(\mathbf{V}_h\) with appropriate boundary conditions.

\[
\mathbf{v}_h = \mathbf{w}_h + Q_2 \nabla \mathbf{v}_h,
\]

where \(G_{Q_2}\) is the mass matrix on \(S_h\) and \(G_{\nabla \mathbf{v}_h}\) is the discrete Laplace operator. Then the discrete case of \(\nabla \nabla \mu = 0\), i.e. \((\nabla \nabla \mu, \nabla \mu) = (G_{\nabla \mathbf{v}_h} \mathbf{v}_h, \nabla \mathbf{v}_h) \forall \mathbf{v}_h \in \mathbf{V}_h\) is bounded by \(\chi^2\), which is very small.

3 Smoothing Procedure in Matrix Form
Let \(T_h\) be the matrix of the operator \(Q_2 \nabla \cdot \nabla\). \(A \mu = 0\) is from \((\nabla \mu, \nabla \mu)\), \(B \mu = 0\) is from \((\nabla \mu, \mu)\).
We want to smooth
\[
\begin{pmatrix}
A \\
B
\end{pmatrix}
\]
Then starting from \((u_0, p_0)\):
- Solve (smooth) \(u_1\) from the equation \(T' \mu_1 = g - B \mu_0\) and let \(u_{1,2} = u_1 + T' \mu_2\)
- Solve (smooth) \(p_1\) from the equation \(T' \pi_1 = f - A \mu_{1,2} - B' \mu_0\), and let \(p_1 = p_0 + p_1\).
- Solve (smooth) \(\pi_2\) from the equation \(A \pi_2 = f - A \mu_{1,2} - B' \mu_0\), with dirichlet boundary condition \(\pi_2 = u_{1,2}\) on \(\delta \Omega\). Let \(u_0 = u_{1,2} + \pi_2\).
Notice for all three steps, we are smoothing Laplace-like problems.

4 Numerical Results
A model problem is computed in \(\Omega \times (0, 1)\), with known solutions
\[
u = \begin{pmatrix}
2x(1 - x)^2y(1 - y)^2
\end{pmatrix}, \quad p = x^2 - y^2.
\]
A two-grid method is presented, with the fine grids for pressure are \(h = 1/4, h = 1/8\), \(h = 1/16, h = 1/32\), the fine grids for velocity are \(1/16, 1/32, 1/4, 1/128\) respectively. We display, the error norm of velocity and pressure with true solutions, and the ratio of these norms to their values at the end of previous cycle. Actual size errors from direct solution method is also presented to compare. We use one step pre and post distributive relaxation as smoothing, in each step of distributive relaxation, we use one step jacobi iteration to smooth the Laplace-like problem.