1. For the given function \( g \), a representation in terms of Heaviside functions is:

\[
g(t) = (2t - 1)H(t - \frac{1}{2}) - (2t - 1)H(t - 1) + (-2t + 3)H(t - 1) - (-2t + 3)H(t - 2) + (2t - 5)H(t - 2) - (2t - 5)H(t - \frac{5}{2}) \]

\[= 2(t - \frac{1}{2})H(t - \frac{1}{2}) - 4(t - 1)H(t - 1) + 4(t - 2)H(t - 2) - 2(t - \frac{5}{2})H(t - \frac{5}{2}).\]

The Laplace transform \( G \) can be read from the table, line 13 which is the Shift Theorem:

\[G(s) = \left(2e^{-\frac{1}{2}s} - 4e^{-s} + 4e^{-2s} - 2e^{\frac{5}{2}s}\right)\left(\frac{1}{s^2}\right).
\]

2. The transformed initial value problem is

\[s^2X + \frac{1}{2}sX + X = e^{-s},\]

which can be easily solved:

\[X = \frac{e^{-s}}{s^2 + \frac{1}{2}s + 1} = \sqrt{\frac{16}{15}} \left(\frac{\left(\frac{15}{16}\right) e^{-s}}{(s^2 + \frac{1}{2}s + \frac{16}{15}) + \frac{15}{16}}\right),\]

which we get by completing the square. This can now be rewritten as

\[X = \sqrt{\frac{16}{15}} \left(\frac{\left(\frac{15}{16}\right) e^{-s}}{s - \left(-\frac{1}{4}\right)^2 + \left(\frac{15}{16}\right)^2}\right)\]

Using the Shift Theorem, the inverse transform gives the solution,

\[x(t) = \sqrt{\frac{16}{15}} e^{-\frac{1}{4}(t-1)} \sin \left(\sqrt{\frac{15}{16}} (t - 1)\right),\]

which asymptotically approaches 0 as \( t \to \infty \) since the sine function is bounded.
3. Applying the Laplace transform to the equation with the convolution product, \( x + (t \ast x) = \sin (2t) \), we get:

\[
X + \left( \frac{1}{s^2} \right) X = \frac{2}{s^2 + 4}.
\]

This equation can be solved for \( X \), which results in:

\[
X = \frac{2s^2}{(s^2 + 1)(s^2 + 4)} = -\frac{2}{3} \frac{1}{s^2 + 1} + \frac{4}{3} \frac{2}{s^2 + 4},
\]

and the inverse transform is then,

\[
x(t) = -\frac{2}{3} \cos (t) + \frac{4}{3} \cos (2t).
\]

4. The partial differential equation is

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < 1, \quad -\pi < \theta \leq \pi.
\]

Separation of variables assuming \( u(r, \theta) = R(r) \Theta(\theta) \) leads to the form

\[
R'' \Theta + \frac{1}{r} R' \Theta + \frac{1}{r^2} R \Theta'' = 0.
\]

Dividing both sides by \( R \Theta \) we arrive at the equation

\[
\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} = 0.
\]

Rearranging and introducing the separation constant \( \lambda^2 \) we arrive at

\[
\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -\frac{1}{r^2} \frac{\Theta''}{\Theta} = \lambda^2.
\]

or

\[
r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2.
\]

Hence the two ordinary differential equations are

\[
r^2 R'' + r R' - \lambda^2 R = 0 \quad \text{and} \quad \Theta'' + \lambda^2 \Theta = 0.
\]

Part (b) is obvious from this result. As for part (c), one reason to use Fourier series is to be able to choose the coefficients in a Fourier representation of the solution as a superposition of eigenfunctions so that the condition \( u(1, \theta) = f(\theta) \) is satisfied.