KAKUTANI’S FIXED POINT THEOREM

**Theorem:** Let \( X \subset \mathbb{R}^n \) be closed, bounded, and convex. For every \( x \in X \) let \( F(x) \) be a non-empty, convex subset of \( X \). Assume that the graph of the set-valued functions is closed in \( X \times X \). Then there exists a point \( x^* \in X \) such that

\[
x^* \in F(x^*).
\]

**Proof:** As with our proof of the Brouwer theorem, we will assume that \( X \) is a simplex with vertices \( v^0, v^1, \ldots, v^n \). Form the \( k^{th} \) simplicial subdivision of \( X \) and define \( f^{(k)} \) as follows:

- if \( x \in \{v^0, v^1, \ldots, v^n\} \), let \( f^{(k)}(x) = y \) where \( y \in F(x) \) (any point will do).
- if \( x \) is any other point of the cell, define \( f^{(k)}(x) \) by interpolation from the values of \( f^{(k)} \) at the vertices. In other words, if \( x = \sum_{j=0}^{n} \theta_j v^j \), then define \( f^{(k)}(x) := \sum_{j=0}^{n} \theta_j f^{(k)}(v^j) \).

Note that if a point lies on a common face of two cells, the definitions are consistent.

Now \( f^{(k)} : X \rightarrow X \) continuously, so by the Brouder theorem, each \( f^{(k)} \) has a fixed point which we call \( x^{(k)} \). Now, suppose that \( x^{(k)} \) lies in a cell of the \( k^{th} \) subdivision \( \langle v^{(k)}_0, v^{(k)}_1, \ldots, v^{(k)}_n \rangle \), and let \( \theta^k_0, \theta^k_1, \ldots, \theta^k_n \) be its barycentric coordinates relative to the cell. Then the fact that \( x^{(k)} = f^{(k)}(x^{(k)}) \) implies that

\[
x^{(k)} = f^{(k)}(x^{(k)}) = \sum_{j=0}^{n} \theta^k_j y^{(k)}_j, \quad \text{where} \quad y^{(k)}_j \in F(x^{(k)}), j = 0, 1, \ldots, n.
\]

Since \( X \) is compact (is closed and bounded) the sequences \( \{x^{(k)}\}_{k=1}^{\infty}, \{\theta^k_j\}_{k=1}^{\infty}, \) and \( \{y^{(k)}_j\}_{k=1}^{\infty}, j = 0, 1, \ldots, n, \) may, after possibly renumbering them, be assumed to converge to points \( x^*, \theta_j, \) and \( y_j, j = 0, 1, \ldots, n \) respectively.

As in the proof of the Brouwer theorem, since the diameter of the cells in successive subdivisions approaches zero as \( k \to \infty \), all the vertices \( v^{(k)}_j \to x^* \) as \( k \to \infty \). Hence

\[
x^{(k)} = f^{(k)}(x^{(k)}) = \sum_{j=0}^{n} \theta_j^k y^{(k)}_j,
\]

implies that \( x^* = \sum_{j=0}^{n} \theta_j y_j \).

Finally, since the set-valued function \( F \) has closed graph, \( y_j \in F(x^*), j = 0, 1, \ldots, n \). Since \( F \) takes convex values, \( F(x^*) \) is convex and so \( x^* \), being a convex combination of the \( y_j \) is in \( F(x^*) \), i.e. \( x^* \in F(x^*) \). Thus \( x^* \) is the required fixed point. \( \blacksquare \)