Some Examples of the use of Green’s Theorem

1 Simple Applications

Example 1.1 Evaluate the line integral
\[ \oint_C (3x - y) \, dx + (x + 5y) \, dy, \text{ where } C := x^2 + y^2 = 1 \]

(1) Method I

Just DO it! Use the obvious parameterization \( x = \cos(t), \ y = \sin(t) \) and write:

\[
\begin{align*}
\int_C (3x - y) \, dx + (x + 5y) \, dy &= \int_0^{2\pi} [(3 \cos(t) - \sin(t))(-\sin(t)) + (\cos(t) + 5\sin(t)) \cos(t)] \, dt \\
&= \int_0^{2\pi} [2\sin(t) \cos(t) + 1] \, dt = \left[ \sin^2(t) + t \right]_0^{2\pi} = 2\pi.
\end{align*}
\]

(2) Method II

Use Green’s Theorem:

\[
\begin{align*}
P &= 3x - y, \quad \frac{\partial P}{\partial y}(x, y) = -1 \\
Q &= x + 5y, \quad \frac{\partial Q}{\partial x}(x, y) = 1
\end{align*}
\]

Hence

\[
\begin{align*}
\oint_C (3x - y) \, dx + (x + 5y) \, dy &= \int_D \int \left[ 1 - (-1) \right] \, d(xy) \\
&= 2 \int_D \int d(x, y) \\
&= 2 \times \text{ area of unit disk} = 2\pi.
\end{align*}
\]
Example 1.2 Compute

\[ I = \oint_C \left[ \left( 2y + \sqrt{1+x^5} \right) \, dx + \left( 5x - e^{y^2} \right) \right] \, dy, \text{ where } C := x^2 + y^2 = 4. \]

This computation would be impossible to do directly; just think about the term \( e^{y^2} \)! But use of Green’s theorem makes it easy:

\[
P = 2y + \sqrt{1+x^5}, \quad \frac{\partial P(x,y)}{\partial y} = 2
\]

\[
Q = 5x - e^{y^2}, \quad \frac{\partial Q(x,y)}{\partial x} = 5
\]

Hence

\[
I = \int \int_D (5 - 2) \, d(xy) = 3 \times A = 3(4\pi) = 12\pi.
\]

REMARK: You should note the general result:

\[
A(D) = \frac{1}{2} \int \int_D d(xy) = \int_C -y \, dx + x \, dy.
\]

This gives us a simple method for computing certain areas.

Example 1.3 Find the area of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

Parameterize the ellipse by \( x = a \cos(t), \ y = b \sin(t) \quad 0 \leq t \leq 2\pi \). Then the area enclosed by the ellipse is

\[
A = \frac{1}{2} \int_0^{2\pi} \left[ (-b \sin(t)) (-a \sin(t)) + (a \cos(t)) (b \cos(t)) \right] \, dt
\]

\[
= \frac{1}{2} \int_0^{2\pi} \left[ (ab \sin^2(t)) + (ab \cos^2(t)) \right] \, dt
\]

\[
= \frac{1}{2} \int_0^{2\pi} ab \, dt = \pi ab.
\]