The Gauss-Jordan Method

a quick introduction

We are interested in solving a system of linear algebraic equations in a systematic manner, preferably in a way that can be easily coded for a machine. The best general choice is the Gauss-Jordan procedure which, with certain modifications that must be used to take into account problems arising from specific difficulties in numerical analysis, can be described very easily. Together with a couple of examples and a couple of exercises that you can do by following the given examples, it is easily mastered.

The idea is to start with a system of equations and, by carrying out certain operations on the system, reduce it to an equivalent system whose solution is easily found. It is based on three observations:

1. For a given system, it does not matter in which order the equations are listed;

2. The system remains unchanged if one equation is multiplied on both sides by a non-zero scalar;

3. If we alter replace one equation by the sum of that equation and a scalar multiple of another, then the system is unchanged.

These simple observations allow us to carry out a great simplification for any system of linear algebraic equations.

We give an example, then we will repeat the same example written in matrix form.

Example 1 Consider the system

(a)

\[
\begin{align*}
    x_1 + x_2 + x_3 - x_4 &= 1 \\
    4x_1 + 5x_2 + 5x_3 + 2x_4 &= 0 \\
    x_1 - 2x_2 - 5x_4 &= 5
\end{align*}
\]

Subtract 5× equation 1 from equation 2:
(b) \[
\begin{align*}
x_1 + x_2 + x_3 - x_4 &= 1 \\
-x_1 + 7x_4 &= -5 \\
x_1 - 2x_2 - 5x_4 &= 5
\end{align*}
\]

Add 2× equation 2 to equation 1, and in the same way, add equation 2 to equation 3:

(c) \[
\begin{align*}
x_2 + x_3 + 6x_4 &= -4 \\
-x_1 + 7x_4 &= -5 \\
-2x_2 + 2x_4 &= 0
\end{align*}
\]

Multiply equation 3 by \( \frac{1}{2} \) and likewise add equation 3 to equation 1:

(d) \[
\begin{align*}
x_3 + 7x_4 &= -4 \\
-x_1 + 7x_4 &= -5 \\
-x_2 + x_4 &= 0
\end{align*}
\]

It is now possible to read off the solution. There are three equations in four unknowns, so we would expect that there is one arbitrary constant. In terms of the variable \( x_4 \) the other variables are given by: \( x_1 = 5 + 7x_4 \), \( x_2 = x_4 \), and \( x_3 = -4 - 7x_4 \).

Now, all of the data in the problem is contained in the coefficients of the system. So we could, in the general case, summarize a given system by the augmented matrix:

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & b_n
\end{pmatrix}
\]

We can describe a little more of the procedural steps by means of the following list:
(A) Choose a **Pivot Element**, that is, any non-zero element in the matrix which does **NOT** appear in the last column. Usually for hand computation, we choose a pivot element which is in a column which contains a lot of zeros and/or is ±1.

(B) Carry out the operations described above until the *column* containing the pivot element contains only zeros. Notice that, in doing so, the *row* containing the pivot element remains unchanged.

(C) Choose another *non-zero* pivot element from a row that does not already contain a previously chosen pivot element and proceed as in the preceding step.

(D) Stop when there is no pivot element in any row that does not contain a previously chosen pivot element.

The number of pivot elements is called the **RANK** of the homogeneous system. It is always true that \( r \leq m \) and \( r \leq n \) since different pivot elements appear in different rows and different columns.

As an example, we go through the algorithm, using the matrix of coefficients that come from Example 1:

**Example 2** Notice that here, \( m < n \) and so we expect that the rank of the matrix will be \( \leq m \).

\[
\begin{pmatrix}
1 & 1 & 1 & -1 & 1 \\
4 & 5 & 5 & 2 & 0 \\
1 & -2 & 0 & -5 & 5
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & -1 & 1 \\
-1 & 0 & 0 & 7 & -5 \\
1 & -2 & 0 & -5 & 5
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 1 & 1 & 6 & -4 \\
-1 & 0 & 0 & 7 & -5 \\
0 & -2 & 0 & 2 & 0
\end{pmatrix}
\]
At this stage we are finished, since each row contains a pivot element. Since
the first three rows contain a pivot element, we say that the variables \( x_1, x_2, \)
and \( x_3 \) are the dependent variables and that \( x_4 \) is the independent (or free)
variable. Clearly, the rank of the augmented matrix is 3 and the set of
solutions can be described as

\[
\mathcal{L} = \{(5 + 7x_4, x_4, -4 - 7x_4, x_4)^\top \in \mathbb{R}^4\}
\]
\[
= (5, 0, -4, 0)^\top + \alpha (7, 1, -7, 1)^\top,
\]
where \( \alpha \) is an arbitrary constant.