1. Consider the spatial epidemic model posed in class. Suppose that instead of diffusing, we assume that the susceptibles undergo logistic growth. In other words, we replace the $\tilde{S}$ equation with the following:

$$\frac{\partial \tilde{S}}{\partial t} = -r \tilde{I} \tilde{S} + b \tilde{S} \left( 1 - \frac{\tilde{S}}{S_0} \right), \quad b > 0.$$  

(a) (3 points) Use the scalings given in class to show that the new equation for $S$ becomes

$$\frac{\partial S}{\partial t} = -IS + \delta S(1 - S), \quad (9.1)$$

and give an expression for $\delta$.

Solution. Letting

$$\tilde{S}(\tilde{t}) = S_0 S(t), \quad \tilde{I}(\tilde{t}) = I_0 I(t), \quad \tilde{t} = t \frac{r}{rS_0},$$

we have

$$S_0(rS_0) \frac{\partial S}{\partial t} = -rS_0^2 IS + bS_0 S(1 - S)$$

$$\frac{\partial S}{\partial t} = -IS + \delta S(1 - S), \quad \delta = \frac{b}{rS_0}.$$  

(b) (2 points) Use (9.1) to explain why if there are initially no infectives, $S$ will approach 1.

Solution. In the case that $I = 0$, (9.1) becomes

$$\frac{\partial S}{\partial t} = \delta S(1 - S).$$

There are two steady states: $S = 0$ and $S = 1$. Since $\partial S/\partial t < 0$ for $S > 1$, we see that $S = 1$ is stable and hence the population will approach this value.

(c) (6 points) Calculate the values of $S$ and $I$ ahead of and behind any traveling wave.

Solution. We let

$$z = x - Vt, \quad S(x,t) = u(z), \quad I(x,t) = v(z)$$
in the *I*-equation from class to yield

\[ v' = y, \quad y' = -cy - v(u - \lambda). \quad (C.1) \]

Then making these substitutions into (9.1), we have

\[ -cu' = -uv + \delta u(1 - u). \quad (C.3) \]

Solving for the steady states, we have from (C.1) that \( y = 0 \). Then from (C.2) we have that \( v = 0 \) or \( u = \lambda \). If \( v = 0 \), we have from (C.3) that \( u = 0 \) or \( u = 1 \). If \( u = \lambda \), then from (C.3) we have that \( v = \delta(1 - \lambda) \). Therefore, the steady states are given by

\[ x_1 = (u, v, y) = (0, 0, 0), \quad x_2 = (1, 0, 0), \quad x_3 = (\lambda, \delta(1 - \lambda), 0). \]

We already know that \( x_2 \) holds ahead of the front by our answer to (b). We also suspect that \( x_1 \) would not hold behind the front, since we expect that if \( v = 0 \), the population \( u \) of the susceptibles would rebound up to 1. To check this, we note that if \( u = 0 \) in (C.1) and (C.2), we have that the Jacobian and its eigenvalues \( \sigma \) are given by

\[ J(x_1) = \begin{pmatrix} 0 & 1 \\ \lambda & -c \end{pmatrix} \implies \sigma = \frac{-c \pm \sqrt{c^2 + 4\lambda}}{2}. \]

Thus we see that the origin is a saddle. But the value behind the front corresponds to \( z \to -\infty \), so we want point which is *unstable* to all trajectories, namely a stable node or spiral. Thus behind the front we have the steady state \( x_3 \).

(d) (2 points) Show that \( \lambda \leq 1 \). What is the minimum wave speed?

**Solution.** We note that (C.1) and (C.2) are exactly those equations given in class. Thus linearizing about the steady state \( (1, 0, 0) \), we again have that the eigenvalues are given by

\[ \sigma = \frac{-c \pm \sqrt{c^2 - 4(1 - \lambda)}}{2}, \]

which implies that \( \lambda \leq 1 \) and \( c \geq 2\sqrt{1 - \lambda} \) for a traveling wave to exist. Hence the minimum wave speed is \( c = 2\sqrt{1 - \lambda} \).

2. (10 points) Optimize the following functional:

\[ J(y, z) = \int_0^1 2y'z' - 4(y'')^2 + z^2 \, dx \]

subject to

\[ y(0) = -1, \quad y(1) = -3, \quad z(0) \text{ unknown}, \quad y'(1) \text{ unknown}. \]
Solution. Since several of the boundary conditions are unknown, we perform the integration by parts to find the natural boundary conditions:

\[ \delta J = \int_0^1 h_y \frac{\partial F}{\partial y} + h_y' \frac{\partial F}{\partial y'} + h_y'' \frac{\partial F}{\partial y''} + h_z \frac{\partial F}{\partial z} + h_z' \frac{\partial F}{\partial z'} \, dx \]

\[ = \int_0^1 h_y \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y''} \right) \, dx + \left[ h_y \frac{\partial F}{\partial y'} + h_y' \frac{\partial F}{\partial y''} \right]_0^1 \]

\[ + \int_0^1 h_z \left( \frac{\partial F}{\partial z} - \frac{d}{dx} \frac{\partial F}{\partial z'} \right) \, dx + \left[ h_z \frac{\partial F}{\partial z'} \right]_0^1. \]

Setting the integrand part of the variation equal to zero, we have

\[ \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y''} = 0 \]

\[ 0 - (2z')' + (-8y'')'' = 0 \]

\[ z'' + 4y^{(4)} = 0. \quad \text{(B.1)} \]

\[ \frac{\partial F}{\partial z} - \frac{d}{dx} \frac{\partial F}{\partial z'} = 0 \]

\[ 2z - (2y')' = 0 \]

\[ z = y''. \quad \text{(B.2)} \]

Substituting these results into \( \delta J \), we have

\[ \delta J = [h_y'(-8y'')]_0^1 + [h_z y']_0^1 = -8[h_y'z + h_z y']_0^1, \]

where we have used the fact that \( h_y = 0 \) at the boundaries. Since \( z(0) \) is unknown, we see that \( h_z(0) \) cannot be specified, so

\[ y'(0) = 0. \quad \text{(C.1)} \]

Since \( y'(0) \) has been specified we have that \( h_y'(0) = 0 \), so

\[ \delta J = -8 \left. h_y'z + h_z y' \right|_{z=1}. \]

Since \( y'(1) \) is unknown, we see that \( h_y'(0) \) cannot be specified, so

\[ z(1) = y''(1) = 0. \quad \text{(C.2)} \]

Since \( z(1) \) has been specified we have that \( h_z(1) = 0 \), so the variation has been set to zero. Combining equations (B), we obtain

\[ y^{(4)} + 4y^{(4)} = 0 \]

\[ y = c_1 + c_2 x + c_3 x^2 + c_4 x^3. \]
Using the boundary conditions, we obtain

\[ y(0) = c_1 = -1, \]
\[ y'(0) = c_2 = 0, \]
\[ y(1) = -1 + c_3 + c_4 = -3 \]
\[ z(1) = y''(1) = 2c_3 + 6c_4 = 0. \]

Solving the last two equations together, we have \( c_3 = -3, \ c_4 = 1 \), so we have

\[ y(x) = x^3 - 3x^2 - 1, \]
\[ z(x) = y'' = 6x - 6. \]

3. Consider an elastic rope of constant density \( \rho \) and variable cross-sectional area \( \tilde{A}(\tilde{x}) \), which at equilibrium occupies the range \( 0 \leq \tilde{x} \leq L \). Fix it at the end \( \tilde{x} = 0 \), and place an object of weight \( W \) at \( \tilde{x} = L \), which will elongate the rope.

(a) (3 points) Show that the force exerted on the rope at position \( \tilde{x} \) is given by

\[ \tilde{F}(\tilde{x}) = W + \rho g \int_{\tilde{x}}^{L} \tilde{A}(\tilde{z}) d\tilde{z}. \]  

\textit{Solution.} The force on the rope at position \( \tilde{x} \) is given by the weight of the mass plus the weight of the rope underneath it, which is given by

\[ \rho g \int_{\tilde{x}}^{L} \tilde{A}(\tilde{z}) d\tilde{z}, \]

since the cross-sectional area varies.

(b) (6 points) The strain \( d\tilde{y}/d\tilde{x} \) at any point on the rope is given by

\[ \frac{d\tilde{y}}{d\tilde{x}} = \frac{\tilde{F}(\tilde{x})}{E\tilde{A}(\tilde{x})}, \]  

where \( E \) is Young’s modulus. Introduce appropriate scalings to obtain the following dimensionless equation:

\[ y(x) = \int_{0}^{x} \left( 1 + \int_{\xi}^{1} A(z) dz \right) \frac{d\xi}{A(\xi)} d\xi. \]

\textit{Solution.} Substituting (9.2) into (9.3), we have

\[ \frac{d\tilde{y}}{d\tilde{x}} = \frac{W + \rho g \int_{\tilde{x}}^{L} \tilde{A}(\tilde{x}) d\tilde{x}}{E\tilde{A}(\tilde{x})}. \]
We let 
\[ \tilde{x} = Lx, \quad \tilde{A}(\tilde{x}) = \frac{W}{\rho g L} A(x), \quad \tilde{y}(\tilde{x}) = y_c y(x), \]
where the first two scalings are suggested by (9.4) and \( y_c \) is to be determined. Substituting our scalings into (A), we have
\[
\begin{align*}
\frac{y_c}{L} \frac{dy}{dx} &= \frac{1}{E W A(x)/\rho g L} \left[ W + \rho g \int_x^1 W A(z) (L dz) \right] \\
y_c \frac{dy}{dx} &= \frac{W L}{E W A(x)/\rho g L} \left[ 1 + \int_x^1 A(z) dz \right] \quad \Rightarrow \quad y_c = \frac{L^2 \rho g}{E} \\
y &= \int_0^x \frac{1}{A(\xi)} \left[ 1 + \int_{\xi}^1 A(z) dz \right] d\xi,
\end{align*}
\]
as required.

(c) (3 points) Suppose that the total (normalized) volume of the rope is given by \( V \), and we want to minimize the elongation of the rope at the end with the weight. Define a function \( B(x) \) such that the problem becomes
\[
\text{minimize } J(B) = -\int_0^1 B \frac{B'}{B'} dx,
\]
and give the proper boundary conditions on \( B \).

Solution. Motivated by the form of the numerator in (9.4), we let
\[
B(x) = 1 + \int_x^1 A(z) dz \quad \Rightarrow \quad A(x) = -B',
\]
so the problem has been established once we realize that we want to minimize \( y(1) \), which accounts for the upper limit in the integral. For boundary conditions, we note that the normalized volume of the rope is
\[
V = \int_0^1 A(z) = B(0) - 1,
\]
so we have \( B(0) = 1 + V, \; B(1) = 1 \).

(d) (5 points) Calculate \( B \) for the optimal rope. Show that the strain is constant for the optimal rope.

Solution. Since
\[
F(B, B') = \frac{B}{B'}
\]
does not depend on \( x \), we see that there is a first integral for this system:
\[
F - B' \frac{\partial F}{\partial B'} = C
\]
\[
\frac{B}{B'} - B' \left[ -\frac{B}{(B')^2} \right] = C
\]
\[
2B = CB'
\]
\[
B = \exp \left( \frac{2(x - 1)}{C} \right),
\]
where we have used the boundary condition at \( x = 1 \). Then solving the boundary condition at \( x = 0 \), we have

\[
B(0) = 1 + V = \exp \left( -\frac{2}{C} \right)
\]

\[
\log(1 + V) = -\frac{2}{C}
\]

\[
C = -\frac{2}{\log(1 + V)}
\]

\[
B(x) = \exp \left( (1 - x) \log(1 + V) \right) = (1 + V)^{1-x}.
\]

\[
\frac{dy}{dx} = -\frac{B}{B'} = -\frac{(1 + V)^{x-1}}{-\log(1 + V)(1 + V)^{x-1}} = \frac{1}{\log(1 + V)}
\]

which is constant, as required.