Homework Set 5 Solutions

1. (6 points) By direct calculation, verify that the integral representation for $J_0(z)$ does indeed solve Bessel’s equation of order 0.

Solution. Substituting the integral representation

$$J_0(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \phi) \, d\phi$$

into Bessel’s equation, we obtain

$$z^2 \frac{d^2}{dz^2} \left[ \frac{1}{\pi} \int_0^\pi \cos(z \sin \phi) \, d\phi \right] + z \frac{d}{dz} \left[ \frac{1}{\pi} \int_0^\pi \cos(z \sin \phi) \, d\phi \right] + z^2 \frac{1}{\pi} \int_0^\pi \cos(z \sin \phi) \, d\phi = 0$$

$$\int_0^\pi \frac{d^2}{dz^2} \left[ \cos(z \sin \phi) \right] + \frac{d}{dz} \left[ \cos(z \sin \phi) \right] + z^2 \cos(z \sin \phi) \, d\phi = 0$$

$$\int_0^\pi \left( z^2 - z^2 \sin^2 \phi \right) \cos(z \sin \phi) \, d\phi - z \sin \phi \sin(z \sin \phi) \, d\phi = 0$$

$$\int_0^\pi \frac{z \cos \phi}{\sin(z \sin \phi)} \, d\phi - z \sin \phi \sin(z \sin \phi) \, d\phi = 0$$

$$\left[ (z \cos \phi) \sin(z \sin \phi) \right]_0^\pi + \int_0^\pi \sin(z \sin \phi) \, d\phi - z \sin \phi \sin(z \sin \phi) \, d\phi = 0$$

$$0 = 0.$$

2. (8 points) Use analogous arguments to those in Theorem 1 on page 308 of Guenther and Lee to prove uniqueness of the solution to the following problem:

$$\nabla^2 u - k^2 u = 0, \quad x \in D \subset \mathbb{R}^3; \quad u(x) = f(x), \quad x \in \partial D. \quad (5.1)$$

Solution. Let $u_1$ and $u_2$ be two solutions of (5.1). Then $\phi = u_1 - u_2$ satisfies

$$\nabla^2 \phi - k^2 \phi = 0, \quad x \in D \subset \mathbb{R}^3; \quad \phi(x) = 0, \quad x \in \partial D. \quad (A)$$

We consider the following integral:

$$J = \iiint_D \phi(\nabla^2 \phi - k^2 \phi) \, dx = \iint_D \nabla \cdot (\phi \nabla \phi) - |\nabla \phi|^2 - k^2 \phi^2 \, dx$$

$$= \int_{\partial D} \phi \nabla \phi \, dS - \iint_D |\nabla \phi|^2 + k^2 \phi^2 \, dx = - \iint_D |\nabla \phi|^2 + k^2 \phi^2 \, dx,$$
where we have used the boundary condition in (A). From the equation in (A), we have
that \(J = 0\), so \(\phi = 0\) and \(u_1 = u_2\).

3. (10 points) A semicircle of radius 1 is insulated along the flat side and has an
imposed temperature of \(f(\theta)\) at the outer boundary (see figure). Show that the
solution to Laplace’s equation in this geometry is given by

\[
T(r, \theta) = \frac{1 - r^2}{\pi} \int_0^\pi \frac{f(\phi)(1 + r^2 - 2r \cos \theta \cos \phi)}{(1 + r^2)^2 - 4r(1 + r^2) \cos \theta \cos \phi + 2r^2(\cos 2\theta + \cos 2\phi)} \, d\phi
\]

\[
T(r, \theta) = \frac{1 - r^2}{\pi} \int_0^\pi \frac{f(\phi)(1 + r^2 - 2r \cos \theta \cos \phi)}{(1 + r^2)^2 - 4r(1 + r^2) \cos \theta \cos \phi + 2r^2(\cos 2\theta + \cos 2\phi)} \, d\phi
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T(r, \theta) = \frac{1 - r^2}{\pi} \int_0^\pi \frac{f(\phi)(1 + r^2 - 2r \cos \theta \cos \phi)}{(1 + r^2)^2 - 4r(1 + r^2) \cos \theta \cos \phi + 2r^2(\cos 2\theta + \cos 2\phi)} \, d\phi
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\]

\[
T(r, \theta) = \frac{1 - r^2}{\pi} \int_0^\pi \frac{f(\phi)(1 + r^2 - 2r \cos \theta \cos \phi)}{(1 + r^2)^2 - 4r(1 + r^2) \cos \theta \cos \phi + 2r^2(\cos 2\theta + \cos 2\phi)} \, d\phi
\]
4. Consider the dimensionless wave equation for the function $\Psi(r, \theta, \phi, t)$ in a sphere of radius 1. In addition, suppose that on the surface $\Psi$ oscillates in the following manner:

$$\Psi(1, \theta, \phi, t) = e^{i\omega t} f(\theta, \phi).$$  \hfill (5.2)

(a) (2 points) Make an appropriate variable substitution to obtain the following governing equation for the function $\psi$:

$$\frac{1}{r^2} \left[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial \psi}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 \psi}{\partial \theta^2} \right] + k^2 \psi = 0,$$  \hfill (5.3)

and calculate the value of $k$. (This is called the Helmholtz equation.) Also calculate the boundary condition at $r = 1$.

Solution. We note from (5.2) that we should let $\Psi(r, \theta, \phi, t) = e^{i\omega t} \psi(r, \theta, \phi)$. Then substituting into the general form of the multidimensional wave equation, we have

$$\nabla^2 (e^{i\omega t} \psi) = c^2 \frac{\partial^2 (e^{i\omega t} \psi)}{\partial t^2} \rightarrow \nabla^2 \psi = -c^2 \omega^2 \psi;$$

$$\nabla^2 \psi + k^2 \psi = 0, \quad k^2 = c^2 \omega^2.$$

By substituting the spherical form of the Laplacian into the above, we obtain (5.3). Clearly we have that

$$\psi(1, \theta, \phi) = f(\theta, \phi).$$

(b) (9 points) Show that the appropriate radial eigenfunctions are given by

$$R_n(r) = \frac{J_{n+1/2}(kr)}{r^{1/2}}, \quad n > 0.$$  \hfill (5.4)

Solution. Separating variables by letting $\psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$, we have

$$\frac{1}{r^2} \left[ \Theta \phi \frac{d(r^2 \Phi')}{dr} + R \Theta \phi \frac{d(\sin \phi \Phi')}{d\phi} + \frac{R \Phi \Theta''}{\sin^2 \phi} \right] + k^2 R \Theta \Phi = 0$$

$$\frac{\sin^2 \phi d(r^2 R')}{R dr} + \frac{\sin \phi d(\sin \phi \Phi')}{\Phi d\phi} + \frac{\Theta''}{\Theta} + k^2 r^2 \sin^2 \phi = 0.$$

Again we see that $\Theta''/\Theta$ must be a constant, and the same arguments as in class imply that

$$\Theta(\theta) = \sin m\theta, \cos m\theta \quad \Rightarrow \quad \frac{\Theta''}{\Theta} = -m^2.$$  \hfill (B.1)

Substituting this result into our equation, we have

$$\frac{1}{R} \frac{d(r^2 R')}{dr} + k^2 r^2 = -\left[ \frac{1}{\sin \phi \Phi} \frac{d(\sin \phi \Phi')}{d\phi} - \frac{m^2}{\sin^2 \phi} \right].$$
The right-hand side is the same as in class, so we have that
\[ \Phi(\phi) = P_n^m(\cos \phi), \] (B.2)
and the right-hand side is \( n(n+1) \), so we have
\[ \frac{d(r^2 R')}{dr} + k^2 r^2 R = n(n+1)R. \]
Letting
\[ R(r) = \frac{Z(z)}{r^{-1/2}}, \quad z = kr \]
in the above, we have
\[ k^2 \frac{d}{dz} \left( \frac{Z(z)}{r^{1/2}} \right) + \frac{z^2}{r^{1/2}} Z = \frac{n(n+1)Z}{r^{1/2}} \]
\[ k^{1/2} \frac{d}{dz} \left( z \left( \frac{Z'}{z^{1/2}} - \frac{Z}{2z^{3/2}} \right) \right) + \frac{k^{1/2}}{z^{1/2}} \left[ z^2 - n(n+1) \right] Z = 0 \]
\[ z^{3/2} Z'' + \frac{3z^{1/2}}{2} Z' - z^{1/2} Z' - \frac{1}{4z^{1/2}} Z + \frac{1}{z^{1/2}} \left[ z^2 - n(n+1) \right] Z = 0 \]
\[ z^2 Z'' + z Z' + \left( n + \frac{1}{2} \right)^2 Z = 0 \]
\[ Z(z) = J_{n+1/2}(z) \]
\[ R_n(r) = \frac{J_{n+1/2}(kr)}{r^{1/2}}, \quad n \geq 0. \]
Here the restriction on \( n \) is due to the fact that \( J_{n+1/2} = O(r^{n+1/2}) \) as \( r \to 0 \), so these are the only ones that are bounded at the origin.

(c) (5 points) Write the solution \( \Psi \) to the problem.

**Solution.** Using (5.4) and (B) in our sum, we have
\[ \psi(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} J_{n+1/2}(kr) \frac{P_n^m(\cos \phi)}{r^{1/2}} [a_{mn} \sin m \theta + b_{mn} \cos m \theta]. \]
Examining the boundary condition, we have
\[ \psi(1, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} J_{n+1/2}(k) P_n^m(\cos \phi) [a_{mn} \sin m \theta + b_{mn} \cos m \theta] \]
\[ f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} J_{n+1/2}(k) P_n^m(\cos \phi) [a_{mn} \sin m \theta + b_{mn} \cos m \theta]. \]
To determine the constants, we realize that since the \( \theta \)- and \( \phi \)-functions are the same as in class, with the only difference being the additional factor \( J_{n+1/2}(k) \). Thus we have that
\[ a_{mn} = \frac{1}{\pi J_{n+1/2}(k)||P_n^m(\cos \phi)||^2} \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) P_n^m(\cos \phi) \sin m \theta \sin \phi \, d\phi \, d\theta, \]
\[ b_{mn} = \frac{1}{\pi J_{n+1/2}(k)||P_n^m(\cos \phi)||^2} \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) P_n^m(\cos \phi) \cos m \theta \sin \phi \, d\phi \, d\theta. \]