Homework Set 4 Solutions

1. (4 points) Use d’Alembert’s formula to solve

\[ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = \delta(x - \xi) \]

in the form

\[ u(x,t) = \frac{H(x + ct) - H(x - ct)}{2c}. \]

Solution. Plugging into d’Alembert’s formula, we have

\[ u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \delta(y - \xi) \, dy = \begin{cases} 
1/2c, & x - ct < \xi < x + ct, \\
0, & \text{else}.
\end{cases} \]

\[ = \begin{cases} 
1/2c, & \xi - ct < x < \xi + ct, \\
0, & \text{else}.
\end{cases} \]

If \( x < \xi - ct \), we note that both \( H(x-(\xi-ct)) \) and \( H(x-(\xi-ct)) \) are zero, so their difference is zero. Similarly, if \( x > \xi - ct \), we note that both \( H(x-(\xi-ct)) \) and \( H(x-(\xi-ct)) \) are one, so their difference is zero. Their difference is nonzero in the interval where \( u \) is nonzero, so we have

\[ u(x,t) = \frac{H(x - \xi + ct) - H(x - \xi - ct)}{2c}. \]

2. Consider the following problem:

\[ \frac{\partial^2 \psi}{\partial t^2} + 2k \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}, \quad k > 0, \quad x \in [0,1]; \quad \frac{\partial \psi}{\partial x}(0,t) = \psi(1,t) = 0. \quad (4.1a) \]

(a) (9 points) Obtain the “normal modes” for (4.1a). Explain how their behavior differs from those of true normal modes. Be sure to consider what happens to the normal modes as \( k \) increases. Don’t miss any special values.

Solution. Separating variables by letting \( \psi(x,t) = \phi_n(x)T_n(t) \), we obtain

\[ \phi_n T_n'' + 2k \phi_n T_n' = \phi_n'' T_n, \]

\[ \phi_n'' = \frac{T_n'' + kT_n'}{T_n} = -\lambda_n^2. \]

\[ \phi_n'' + \lambda_n^2 \phi_n = 0, \quad (A.1) \]

\[ T_n'' + 2k T_n' + \lambda_n^2 T_n = 0. \quad (A.2) \]
where the right-hand side has been chosen for simplicity. Since the boundary conditions for (A.1) are given by
\[ \phi_n'(0) = \phi_n(1) = 0, \]
we see that the solution to (A.1) and the boundary condition at \( x = 0 \) is given by \( \cos \lambda_n x \).

Then satisfying the boundary condition at \( x = 1 \), we have
\[ -\lambda_n \sin \lambda_n = 0 \implies \lambda_n = \left( n + \frac{1}{2} \right) \pi \]
\[ \phi_n(x) = \cos \left( n + \frac{1}{2} \right) \pi x, \quad n > 0. \]

Making this substitution into (A.2), we obtain
\[ T'' + 2kT' + \left( n + \frac{1}{2} \right) \pi^2 T = 0 \]
\[ T_n(t) = e^{\alpha_n t} \implies \alpha_n^2 + 2k\alpha_n + \left( n + \frac{1}{2} \right) \pi^2 = 0 \]
\[ \alpha_n = -k \pm \sqrt{\left( n + \frac{1}{2} \right) \pi^2 - k^2}, \]
\[ T_n(t) = e^{-kt} [c_1 \sin(\omega_n t) + c_2 \cos(\omega_n t)], \quad (B.1) \]
\[ \omega_n = \sqrt{\left( n + \frac{1}{2} \right) \pi^2 - k^2}. \quad (B.2) \]

Therefore, the “normal modes” are given by
\[ e^{-kt} \cos \left( \left( n + \frac{1}{2} \right) \pi x \right) \sin(\omega_n(t - t_n)). \]

These are not true normal modes in that all solutions decay as \( t \to \infty \). Also, we note that as \( k \) increases above \( n\pi \), then \( \omega_j \) is imaginary for \( j \leq n \). Thus those modes become pure exponentials of the following form:
\[ e^{-kt} \cos \left( \left( n + \frac{1}{2} \right) \pi x \right) \sinh(r_n(t - t_n)), \quad r_n = \sqrt{k^2 - \left( n + \frac{1}{2} \right) \pi^2}. \]

Thus these modes do not oscillate with time but decay exponentially fast (since \( r_n < k \)). In addition, if \( k = (n+1/2)\pi \), we have a double root of \( \alpha_n = -kt \), which leads to a solution
\[ T_n(t) = e^{-kt}(c_1 + c_2t). \]
(b) (2 points) Explain why (4.1a) models wave propagation with damping.

_solution_. Since the oscillations always decay to zero as \( t \to \infty \), this equation models a damped physical process.

(c) (4 points) Show that the solution to (4.1a) with

\[ \psi(x,0) = f(x), \quad \frac{\partial \psi}{\partial t}(x,0) = g(x) \] \hspace{1cm} (4.1b)

is usually given by

\[ \psi(x,t) = \sum_{n=0}^{\infty} e^{-kt} \cos \left( \left( n + \frac{1}{2} \right) \pi x \right) \left[ f_n \cos(\omega_n t) + \frac{g_n + kf_n}{\omega_n} \sin(\omega_n t) \right], \]

and write the expressions for \( f_n \), \( g_n \), and \( \omega_n \).

_solution_. Then by standard Sturm-Liouville theory, we know that the boundary conditions for (A.2) are given by

\[ T_n(0) = f_n, \quad T'_n(0) = g_n, \] \hspace{1cm} (C)

where

\[ f_n = \frac{\langle f, \phi_n \rangle}{\| \phi_n \|^2}. \]

Here we interpret “usually” to mean the case where \( k \neq (n + 1/2)\pi \). (Here we include the case where \( \omega_n \) is imaginary.) Calculating the denominator, we have

\[ \| \phi_n^2 \| = \int_0^1 \cos^2 \left( \left( n + \frac{1}{2} \right) \pi x \right) dx = \int_0^1 \frac{1 + \cos((2n + 1)\pi x)}{2} dx \\
= \left[ \frac{x}{2} + \frac{\sin((2n + 1)\pi x)}{4} \right]_0^1 = \frac{1}{2}, \]

\[ f_n = 2 \int_0^1 f(x) \cos \left( \left( n + \frac{1}{2} \right) \pi x \right) dx, \]
\[ g_n = 2 \int_0^1 g(x) \cos \left( \left( n + \frac{1}{2} \right) \pi x \right) dx. \]

Substituting (B.1) into (C), we obtain

\[ T_n(0) = c_2 = f_n \]
\[ T'_n(0) = -kc_2 + c_1 \omega_n = g_n \]
\[ c_1 = \frac{g_n + kf_n}{\omega_n} \]
\[ T_n(t) = e^{-kt} \left[ \frac{g_n + kf_n}{\omega_n} \sin(\omega_n t) + f_n \cos(\omega_n t) \right], \]
from which our result immediately follows.

3. A membrane occupies the quarter-annular region $\tilde{r} \in [r_{\text{in}}, r_{\text{out}}]$, $\theta \in [0, \pi/2]$ (see figure). The membrane is free on the edges of constant $\theta$ and clamped at the edges of constant $r$. Initially the membrane is undisturbed, and is then given a velocity of

$$\frac{\partial \tilde{\psi}}{\partial \tilde{t}}(\tilde{r}, \theta, 0) = \psi_0 \tilde{f}(\tilde{r}).$$

(a) (6 points) Derive the following dimensionless governing equation for the displacement $\psi(r, \theta, t)$:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = \frac{\partial^2 \psi}{\partial \tilde{t}^2}, \quad r \in [1, R], (4.3a)$$

and write the expressions for $f$ and $R$.

Solution. From notes in class, we have that the dimensional wave equation is given by

$$c^2 \left[ \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{\psi}}{\partial \tilde{r}} \right) + \frac{1}{\tilde{r}^2} \frac{\partial^2 \tilde{\psi}}{\partial \tilde{\theta}^2} \right] = \frac{\partial^2 \tilde{\psi}}{\partial \tilde{t}^2}, \quad c^2 = \frac{T_0}{\rho}. \quad (D)$$

Motivated by the right-hand side of (4.2), the domain in (4.3a), and the desire to eliminate $c^2$, we let

$$r = \frac{\tilde{r}}{r_{\text{in}}}, \quad \psi(r, \theta, t) = \frac{\psi(\tilde{r}, \theta, \tilde{t})}{\psi_0}, \quad t = \frac{\tilde{t}}{r_{\text{in}} \sqrt{\frac{T_0}{\rho}}} = \frac{\tilde{t}}{r_{\text{in}}},$$

in (D) to obtain

$$\psi_0 \frac{T_0}{\rho r_{\text{in}}^2} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right] = \psi_0 \frac{T_0}{\rho r_{\text{in}}^2} \frac{\partial^2 \psi}{\partial \tilde{t}^2}, \quad r \in [1, r_{\text{out}}/r_{\text{in}}].$$

Therefore, by letting

$$R = \frac{r_{\text{out}}}{r_{\text{in}}},$$
we obtain (4.3a). Making the substitutions into (4.2), we obtain
\[ \psi_0 \frac{1}{r_{in}} \sqrt{\frac{T_0}{\rho}} \frac{\partial \psi}{\partial t}(r, \theta, 0) = \psi_0 \tilde{f}(\tilde{r}). \]

Therefore, by letting
\[ f(r) = r_{in} \tilde{f}(\tilde{r}) \sqrt{\frac{\rho}{T_0}} = \frac{r_{in} \tilde{f}(\tilde{r})}{c}, \]
(4.3b) immediately follows.

(b) (4 points) Write down the other boundary conditions needed to solve the problem.

**Solution.** The edges of constant \( \theta \) are free, which means that there is no force applied, so we have
\[ \frac{\partial \psi}{\partial \theta}(r, 0, t) = \frac{\partial \psi}{\partial \theta}(r, \pi/2, t) = 0. \]  
(E.1)
The edges at \( r = 1 \) and \( r = R \) are clamped, so we have
\[ \psi(1, \theta, t) = \psi(R, \theta, t) = 0. \]  
(E.2)
Initially the membrane is quiescent, so
\[ \psi(r, \theta, 0) = 0. \]  
(E.3)

(c) (6 points) Show that the spatial eigenfunctions \( \phi_n(r) \) are given by
\[ \phi_n(r) = J_0(\lambda_n)Y_0(\lambda_n) - J_0(\lambda_n)Y_0(\lambda_n), \]
\[ J_0(\lambda_n)Y_0(\lambda_n) = J_0(R\lambda_n)Y_0(\lambda_n). \]

Explain why there is only a one-parameter family of eigenfunctions.

**Solution.** We note from (4.3b) and (E.1) that there is no \( \theta \)-dependence in the problem. Separating variables as in class by letting \( \psi(r, \theta, t) = \phi(r)T(t) \), we have
\[ \frac{1}{r \phi} \frac{d}{dr} \left( r \phi \frac{d \phi'}{dr} \right) \frac{T''}{T} = -\lambda^2 \]
(F)
\[ z^2 \frac{d^2 \phi}{dz^2} + z \frac{d \phi}{dz} + (z^2 - 0^2) \phi = 0, \quad z = r \lambda, \]
\[ \phi(z) = c_1 J_0(z) + c_0 Y_0(z), \quad \phi(r) = c_1 J_0(r \lambda) + c_0 Y_0(r \lambda). \]
The boundary conditions at \( r = 1 \) and \( r = R \) are given by
\[ \phi(1) = 0, \quad \phi(R) = 0. \]
Satisfying these conditions, we have
\[ \phi(1) = \lambda [c_1 J_0(\lambda) + c_2 Y_0(\lambda)] = 0 \]
\[ c_1 = -\frac{c_2 Y_0(\lambda)}{J_0(\lambda)}. \]

Letting \( c_2 = -J_0(\lambda) \) for simplicity, we have
\[ \phi(r) = Y_0(\lambda) J_0(r\lambda) - J_0(\lambda) Y_0(r\lambda) \]
\[ \phi(R) = \lambda [Y_0(\lambda) J_0(R\lambda) - J_0(\lambda) Y_0(R\lambda)] = 0. \]

With simple manipulations, these equations yield the desired result, keeping in mind that there will be an infinite number of eigenvalues which are indexed by \( n \). Since there is no \( \theta \)-dependence, there is no need to index the Bessel functions by their order.

(d) (5 points) Write the solution to the problem.

**Solution.** Solving the \( t \)-dependent part of (F), we have
\[ T(t) = c_3 \sin \lambda_n t + c_4 \cos \lambda_n t. \]

The boundary conditions are given by
\[ T_n(0) = 0, \quad T'_n(0) = f_n, \]
where
\[ f_n = \frac{\langle f, \phi_n \rangle}{||\phi_n||^2}, \quad \langle f, \phi_n \rangle = \int_1^R r f(r) \phi_n(r) \, dr. \]

The solution of the above is
\[ T_n(t) = \frac{f_n}{\lambda_n} \sin \lambda_n t \]
\[ \psi(r, \theta, t) = \sum_{n=0}^{\infty} \frac{f_n}{\lambda_n} (\sin \lambda_n t) \phi_n(r). \]