Supplemental Study Material Solutions

1. Consider the flow and density given by
\[\begin{align*}
x &= a_1 \cos \omega t - a_2 \sin \omega t, \\
y &= a_1 \sin \omega t + a_2 \cos \omega t,
\end{align*}\]
(S.1)
\[
\rho(x, y, t) = \rho_0(x \cos \beta t + y \sin \beta t).
\]

(a) (4 points) Show that the flow in (S.1) is steady. Describe the particle paths.

Solution. The velocity field is given by
\[
\begin{align*}
v_x &= \frac{dx}{dt} = -a_1 \omega \sin \omega t - a_2 \omega \cos \omega t = -\omega y, \\
v_y &= a_1 \omega \cos \omega t - a_2 \omega \sin \omega t = \omega x.
\end{align*}
\]
Since the velocity field can be written in such a way so as not to explicitly depend on \(t\), the flow is steady. Since both \(x\) and \(y\) oscillate, we expect orbits about the origin. To check, we note that
\[
x^2 + y^2 = (a_1 \cos \omega t - a_2 \sin \omega t)^2 + (a_1 \sin \omega t + a_2 \cos \omega t)^2
= a_1^2 + a_2^2 - a_1 \sin 2\omega t + a_1a_2 \sin 2\omega t = a_1^2 + a_2^2.
\]
Therefore, the particle paths are circles about the origin.

(b) (4 points) Calculate \(D\rho/Dt\).

Solution. We have
\[
\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y}
= \rho_0[\beta(-x \sin \beta t + y \cos \beta t) - \omega y(\cos \beta t) + \omega x(\sin \beta t)]
= \rho_0[x(\omega - \beta) \sin \beta t + (\beta - \omega)y \cos \beta t].
\]

(c) (2 points) List conditions under which this flow will satisfy conservation of mass.

Solution. Conservation of mass is given by
\[
\frac{D\rho}{Dt} + \rho \nabla \cdot (v_x, v_y) = 0
\]
\[
\rho_0[x(\omega - \beta) \sin \beta t + (\beta - \omega)y \cos \beta t] + \rho \left[\frac{\partial (-\omega y)}{\partial x} + \frac{\partial (\omega x)}{\partial y}\right] = 0
\]
\[\omega = \beta.\]
2. (In this problem, all variables have dimensions.) The vorticity, which measures the rotation of a fluid, is given by \( \vec{\omega} = \nabla \times \mathbf{u} \). The vorticity equation for \( \vec{\omega} \), which is derived by taking the curl of the momentum equation, is given by

\[
\rho \frac{D\vec{\omega}}{Dt} = \rho(\vec{\omega} \cdot \nabla)\mathbf{u} + \mu \nabla^2 \vec{\omega}.
\] (S.2)

(a) (3 points) Explain physically why the evolution of \( \vec{\omega} \) does not depend on either the pressure or gravity.

Solution. The curl of any vector field measures its rotation. Since gravity is a constant force in a single direction, it would not contribute to rotation. (Mathematically, \( \nabla \times \mathbf{g} = 0 \) when we take the curl of the momentum equation.) The pressure contributes to the momentum only through its gradient. In particular, changes in pressure would tend to cause the fluid to move in one direction, rather than to spin. (Mathematically, \( \nabla \times \nabla p = 0 \) when we take the curl of the momentum equation since the curl of any gradient field is 0.)

(b) (4 points) If the flow is two-dimensional, derive the following scalar equation for the vorticity \( \omega \):

\[
\rho \frac{D\omega}{Dt} = \mu \nabla^2 \omega,
\]

and explain the relationship between \( \omega \) and \( \vec{\omega} \).

Solution. For simplicity, let \( \mathbf{u} = (u_x(x,y), u_y(x,y), 0) \). Then

\[
\vec{\omega} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial/\partial x & \partial/\partial y & \partial/\partial z \\
u_x(x,y) & u_y(x,y) & 0
\end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + \left( \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \mathbf{k}.
\]

Thus, denoting the z-component of \( \vec{\omega} \) by \( \omega \), we see that (S.2) becomes a scalar equation for the z-component of \( \vec{\omega} \). Thus, we see that

\[
(\vec{\omega} \cdot \nabla) = \omega \frac{\partial}{\partial z},
\]

and since \( \mathbf{u} \) does not depend on \( z \), that term in (S.2) vanishes, leaving the desired result.

3. Consider the problem of flow between two infinite plates discussed in class, except now assume that \( U < 0 \).

(a) (3 points) Show that for certain values of \( \alpha \), there exists a stagnation point \( y_s \) where \( u(y_s) = 0 \), and calculate \( y_s(\alpha) \).

Solution. If we let

\[
\alpha = \frac{|\Delta \tilde{P}|L^2}{|U|\mu} = \frac{L^2 \Delta \tilde{P}}{U \mu},
\]

then our solution is

\[
u(y) = \frac{\alpha y(1 - y)}{2} - y,
\]
and hence we have

\[ \alpha \frac{(1 - y_s)}{2} - 1 = 0 \]

\[ y_s = 1 - \frac{2}{\alpha} \]

which is meaningful only for \( \alpha > 2 \).

(b) (2 points) Interpret your result as \( U \to 0^- \).

Solution. As \( U \to 0^- \), \( \alpha \to \infty \), so we have \( y_s \to 1 \). In other words, the pressure-driven flow is so intense that the flow is positive throughout almost the entire domain. Therefore, the only region where the flow is negative is very near \( y = 1 \), and hence we have that \( y_s \approx 1 \).

4. Two spheres of equal radius \( a \) are being pushed towards each other at constant relative velocity \( U \) (see figure on next page). Let the distance between their centers be denoted by \( d(t) \), and furthermore let

\[ \frac{d(t) - 2a}{a} \equiv \epsilon(t) \ll 1. \]

The perpendicular bisector of the line connecting their centers is denoted by \( P \). In addition, cylindrical coordinates are introduced as shown in the figure, where the origin is the intersection of \( P \) and the line connecting the spheres' centers.

(a) (2 points) Explain why for the purposes of this problem we may ignore the variance of \( d \) (and hence \( \epsilon \)) on \( t \) and treat them as parameters.

Solution. Since the relative velocity is constant, the problem is steady and so there are no \( \partial/\partial t \) terms anywhere. Therefore, \( t \) appears as a parameter, which can immediately be folded into the parameters \( d \) and \( \epsilon \).

(b) (2 points) Explain why the problem is invariant under rotations in \( \theta \) and hence has no \( \theta \)-dependence at all. Describe any symmetries in \( \tilde{z} \). In particular, obtain the boundary conditions

\[ \tilde{u}_z(\tilde{r}, \tilde{z} = 0) = 0, \quad \frac{\partial \tilde{u}_r}{\partial \tilde{z}}(\tilde{r}, \tilde{z} = 0) = 0. \] (S.3)

Solution. Since this plane is fixed at the perpendicular bisector, we see that each sphere is moving toward it with velocity \( U/2 \). Therefore, we would expect the problem to have odd symmetry in \( \tilde{u}_z \) and even symmetry in \( \tilde{u}_r \) with respect to \( \tilde{z} \). Since the two objects are spheres centered about the line going through the origin, we would expect to have no variation in \( \theta \) whatsoever.
Here are the constant-density steady Navier-Stokes equations in cylindrical coordinates, neglecting gravity, which you may find helpful:

\[
\begin{align*}
\rho \left( \frac{\partial \tilde{u}_r}{\partial \tilde{r}} + \frac{\tilde{u}_\theta}{\tilde{r}} \frac{\partial \tilde{u}_r}{\partial \tilde{\theta}} - \tilde{u}_\theta \frac{\partial \tilde{u}_r}{\partial \tilde{z}} + \tilde{u}_z \frac{\partial \tilde{u}_r}{\partial \tilde{z}} \right) &= -\frac{\partial \tilde{p}}{\partial \tilde{r}} + \mu \left\{ \frac{\partial}{\partial \tilde{r}} \left[ \frac{1}{\tilde{r}} \frac{\partial (\tilde{r} \tilde{u}_r)}{\partial \tilde{r}} \right] + \frac{1}{\tilde{r}^2} \frac{\partial^2 \tilde{u}_r}{\partial \tilde{\theta}^2} - 2 \frac{\partial \tilde{u}_\theta}{\partial \tilde{\theta}} + \frac{\partial^2 \tilde{u}_\theta}{\partial \tilde{z}^2} \right\}, \\
\rho \left( \frac{\partial \tilde{u}_\theta}{\partial \tilde{r}} + \frac{\tilde{u}_\theta}{\tilde{r}} \frac{\partial \tilde{u}_\theta}{\partial \tilde{\theta}} + \tilde{u}_r \frac{\partial \tilde{u}_\theta}{\partial \tilde{r}} + \tilde{u}_z \frac{\partial \tilde{u}_\theta}{\partial \tilde{z}} \right) &= -\frac{\partial \tilde{p}}{\partial \tilde{\theta}} + \mu \left\{ \frac{\partial}{\partial \tilde{r}} \left[ \frac{1}{\tilde{r}} \frac{\partial (\tilde{r} \tilde{u}_\theta)}{\partial \tilde{r}} \right] + \frac{1}{\tilde{r}^2} \frac{\partial^2 \tilde{u}_\theta}{\partial \tilde{\theta}^2} + 2 \frac{\partial \tilde{u}_r}{\partial \tilde{\theta}} + \frac{\partial^2 \tilde{u}_r}{\partial \tilde{z}^2} \right\}, \\
\rho \left( \frac{\partial \tilde{u}_z}{\partial \tilde{r}} + \frac{\tilde{u}_\theta}{\tilde{r}} \frac{\partial \tilde{u}_z}{\partial \tilde{\theta}} + \tilde{u}_r \frac{\partial \tilde{u}_z}{\partial \tilde{r}} + \tilde{u}_z \frac{\partial \tilde{u}_z}{\partial \tilde{z}} \right) &= -\frac{\partial \tilde{p}}{\partial \tilde{z}} + \mu \left\{ \frac{1}{\tilde{r}} \frac{\partial (\tilde{r} \tilde{u}_z)}{\partial \tilde{r}} + \frac{1}{\tilde{r}^2} \frac{\partial^2 \tilde{u}_z}{\partial \tilde{\theta}^2} + \frac{\partial^2 \tilde{u}_r}{\partial \tilde{z}^2} \right\}.
\end{align*}
\]

(S.4)

Since the problem is symmetric about the plane \( P \), we may restrict our analysis to the range to \( 0 < \tilde{z} < \tilde{h}(\tilde{r}) \), where \( \tilde{h}(\tilde{r}) \) describes the surface of the sphere. Therefore, due to our symmetries, we see that the proper boundary conditions for \( z = 0 \) are the following:

\[
\tilde{u}_z(\tilde{r}, 0) = 0, \quad \frac{\partial \tilde{u}_r}{\partial \tilde{z}}(\tilde{r}, 0) = 0,
\]

as required.
(c) (8 points) Determine the proper scalings needed to obtain the following nondimensionalized boundary conditions:

\[
\begin{align*}
uz(r, z = \frac{1 + r^2}{2} + O(\epsilon)) &= -\frac{1}{2}, \\
ur(r, z = \frac{1 + r^2}{2} + O(\epsilon)) &= 0.
\end{align*}
\]  

(S.6)

Discuss how these scalings support the thin-film approximation.

Solution. Careful examination of the surface of the sphere shows that \( \tilde{h}(\tilde{r}) \) is given by

\[
\left[ \frac{d}{2} - \tilde{h}(\tilde{r}) \right]^2 + \tilde{r}^2 = a^2
\]

\[
\tilde{h}(\tilde{r}) = \frac{d}{2} - \sqrt{a^2 - \tilde{r}^2}.
\]

Therefore, we see that our characteristic height for our interface is given by the value at \( \tilde{r} = 0 \), which is simply half the gap width. We want this to scale to \( 1/2 \), so we let

\[
\tilde{z} = (d - 2a)z = \epsilon az,
\]

so we have

\[
\tilde{h}(\tilde{r}) = \frac{d}{2a\epsilon} - \frac{1}{\epsilon} \sqrt{1 - \frac{\tilde{r}^2}{a^2}}
\]

\[
= \frac{1}{2} + \frac{1}{\epsilon} \sqrt{1 - \frac{\tilde{r}^2}{a^2}}.
\]

Therefore, we see that the only way to get an \( O(1) \) result is for the square root to be very near 1. Therefore, we see that we must scale near \( \tilde{r} = 0 \), and hence we let

\[
\tilde{r} = a\epsilon^n r, \quad n > 0,
\]

which yields

\[
\tilde{h}(\tilde{r}) = \frac{1}{2} + \frac{1}{\epsilon} \sqrt{1 - \epsilon^{2n} r^2}
\]

\[
= \frac{1}{2} + \frac{\epsilon^{2n-1} r^2}{2} + O(\epsilon^{4n-1}).
\]

Therefore, we see that for the richest balance [as well as to match (S.6)], we must take \( n = 1/2 \). This makes sense, since the film is thin only in the neighborhood of their closest points. Then the film becomes thicker, and becomes infinitely thick for \( \tilde{r} > a \).

To derive the rest of (S.6), we note that since we are in the plane of symmetry, each sphere is approaching the plane at half the velocity, so we have

\[
\tilde{u}_z(r, h(r)) = -U/2.
\]
Therefore, we must let
\[ \tilde{u}_z = U u_z, \quad (A.3) \]
and the first equation in (S.6) results. To obtain the scaling for \( u_r \), we substitute our results in (A) and our symmetries into the continuity equation (S.4):
\[
\frac{1}{a \epsilon^{1/2}} \frac{1}{r} \frac{\partial (r \tilde{u}_r)}{\partial r} + \frac{U}{a \epsilon} \frac{\partial u_z}{\partial z} = 0, \quad (B)
\]
which means that we should let
\[ \tilde{u}_r = \frac{U}{\epsilon^{1/2}} u_r, \quad (C) \]
which also makes sense since we expect our tangential velocity to be larger than our radial velocity as the fluid gets squeezed out.

(d) (10 points) Using your answer to (c), show that
\[ p \sim \frac{3}{2(1 + r^2)^2}, \quad (S.7) \]
and solve for \( u_r \). You may assume that \( p(\infty) = 0 \).

**Solution.** Using our symmetries and scalings in (A) and (C), we see that (B) and (S.5) reduce to
\[
\frac{U^2 \rho}{\epsilon^{3/2} a} \left( u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} \right) = -\frac{1}{a \epsilon^{1/2}} \frac{\partial \tilde{p}}{\partial r} + \frac{\mu U}{\epsilon^3 a^2} \left\{ \frac{\partial}{\partial r} \left[ \frac{1}{\epsilon} \frac{\partial (ru_r)}{\partial r} \right] + \frac{1}{\epsilon} \frac{\partial^2 u_r}{\partial z^2} \right\},
\]
\[
\frac{U^2 \rho}{\epsilon a} \left( u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{1}{\epsilon a} \frac{\partial \tilde{p}}{\partial z} + \frac{\mu U}{a^2 \epsilon} \left[ \frac{\partial}{\partial r} \left( \frac{1}{\epsilon} \frac{\partial u_z}{\partial r} \right) + \frac{1}{\epsilon} \frac{\partial^2 u_z}{\partial z^2} \right].
\]
We note that \((6\theta)\) is removed entirely since there is no variation in the \( \theta \) direction.

Keeping the leading-order terms from the above equations leads to two possible balances:
\[
\frac{1}{a \epsilon^{1/2}} \frac{\partial \tilde{p}}{\partial r} = \frac{\mu U}{\epsilon^{5/2} a^2} \frac{\partial^2 u_r}{\partial z^2},
\]
\[
\frac{1}{\epsilon a} \frac{\partial \tilde{p}}{\partial z} = \frac{\mu U}{a^2 \epsilon^2} \frac{\partial^2 u_z}{\partial z^2}.
\]
As in class, we note that if we balance the second equation by letting \( \tilde{p} = O(\epsilon^{-1}) \), we would have a linear profile for \( u_r \) in \( z \), which is inconsistent with our symmetry condition. Therefore, we see that for a balance, we should let
\[ \tilde{p} = \frac{\mu U}{\epsilon^2 a} [p^0 + o(1)], \quad u_r = u_r^0 + o(1), \quad u_z = u_z^0 + o(1), \]
which yields, to leading order in $\epsilon$,

\[
\frac{1}{r} \frac{\partial (ru_r^0)}{\partial r} + \frac{\partial u_z^0}{\partial z} = 0, \tag{D.1}
\]
\[
\frac{\partial p^0}{\partial r} = \frac{\partial^2 u_r^0}{\partial z^2}, \tag{D.2}
\]
\[
\frac{\partial p^0}{\partial z} = 0. \tag{D.3}
\]

These are the standard lubrication equations in cylindrical geometry.

Solving (D.2) subject to (S.6) and the dimensionless version of (S.3) and using (D.3), we have

\[
u_r^0 = \frac{1}{2} \frac{dp^0}{dr} \left[ z^2 - \left( \frac{1 + r^2}{2} \right)^2 \right]. \tag{E.1}
\]

Using (E.1) in (D.1), we have

\[
\frac{\partial u_z^0}{\partial z} = -\frac{1}{2r} \frac{\partial}{\partial r} \left\{ r \frac{dp^0}{dr} \left[ z^2 - \left( \frac{1 + r^2}{2} \right)^2 \right] \right\}
\]
\[
u_z^0 = -\frac{1}{2r} \frac{\partial}{\partial r} \left( r \frac{dp^0}{dr} \right) \left[ \frac{z^3}{3} - \left( \frac{1 + r^2}{2} \right)^2 z \right] + \frac{z dp^0}{2 dr} \frac{\partial}{\partial r} \left[ \left( \frac{1 + r^2}{2} \right)^2 \right],
\]

where we have satisfied the boundary condition at zero. In satisfying the other condition, we use (S.6) to obtain a variant of the Reynolds equation:

\[
-\frac{1}{2} = -\frac{1}{2r} \left\{ -\frac{2}{3} \left( \frac{1 + r^2}{2} \right)^3 \frac{\partial}{\partial r} \left( r \frac{dp^0}{dr} \right) + \frac{2r}{2} \left( \frac{1 + r^2}{2} \right) \frac{dp^0}{dr} \left( \frac{1 + r^2}{2} \right) \right\}
\]
\[
-\frac{3r}{2} = \left( \frac{1 + r^2}{2} \right)^3 \frac{\partial}{\partial r} \left( r \frac{dp^0}{dr} \right) + \left( r \frac{dp^0}{dr} \right) 3r \left( \frac{1 + r^2}{2} \right)^2
\]
\[
-\frac{3r^2}{4} + A = r \frac{dp^0}{dr} \left( \frac{1 + r^2}{2} \right)^3
\]
\[
\frac{dp^0}{dr} = -\frac{6r}{(1 + r^2)^3} + \frac{8A}{r(1 + r^2)^3}.
\]

But the pressure gradient must be bounded at $r = 0$, so we have that $A = 0$ and

\[
p^0 = \frac{3}{2(1 + r^2)^2},
\]
\[
u_r^0 = \frac{-3r}{(1 + r^2)^3} \left[ z^2 - \left( \frac{1 + r^2}{2} \right)^2 \right],
\]
as required.