Homework Set 10 (Revised)

Read sections G11-1, L3.6, W36, and W38.

Variational Principles

1. (10 points) Find extremals of the constrained problem

\[ J(y) = \int_1^e x^2(y')^2 \, dx, \quad y(1) = y(e) = 0, \quad \int_1^e y^2 \, dx = 1. \]

The Rayleigh Quotient

**WARNING**: Be sure to consider the sign of \( \lambda \) in each example before applying the techniques used in the notes.

2. Consider the equation

\[ y'' + 3xy = \lambda y, \quad y(0) = 0, \quad y'(1) = 0. \]

(a) (3 points) Construct the Rayleigh quotient for this problem.

(b) (5 points) Use a quadratic trial function in the Rayleigh quotient to show that \( \lambda_1 \geq -7/16. \)

3. Consider the equation

\[ y'' + \frac{y'}{x} + \lambda y = 0, \quad y(0) \text{ bounded}, \quad y'(1) = 0. \quad (10.1) \]

(a) (4 points) Construct the Rayleigh quotient \( Q(y) \) for this problem. Verify that it is always non-positive. (**Hint: Rewrite in standard Sturm-Liouville form**.)

(b) (3 points) Show that \( \lambda = 0 \) is an eigenvalue for (10.1), and explain why it must be the least eigenvalue.

(c) (5 points) Use a quadratic trial function in the Rayleigh quotient to show that \( j_{1,1} \leq 2\sqrt{30/7}. \) (**Hint: What extra condition must the trial function satisfy?**)

(d) (7 points) Verify that

\[ Y(x; \beta) = \beta(\beta + 1)(1 - x)^{\beta - 1} - 2, \quad \beta > 2, \quad (10.2) \]

is an acceptable trial function to calculate \( j_{1,1} \). Show that we may write the Rayleigh quotient as a function of \( \beta \) alone:

\[ q(\beta) = \frac{\beta^2(\beta + 1)^2(2\beta - 1)}{(2\beta - 3)(\beta - 1)(\beta + 4)}. \quad (10.3) \]

(e) (3 points) Use some type of computer program to optimize the Rayleigh quotient (10.3) numerically. What is the approximate gain in accuracy in your bound on \( j_{1,1} \)?