Series Solutions

Consider the equation

$$y'' - y = 0.$$ 

The solution $y_1$ with $y_1(0) = 1$, $y_1'(0) = 0$ is $\cosh x$, which we showed had a series solution given by

$$y_1 = \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$ 

The first three approximations to the series are then given by

$$y_1 \approx 1,$$ 

$$y_1 \approx 1 + \frac{x^2}{2},$$ 

$$y_1 \approx 1 + \frac{x^2}{2} + \frac{x^4}{24}.$$ 

These approximations are graphed below. Note that with each increasing term, the range of $x$ for which the polynomial is a good approximation widens.

In increasing order of thickness: Polynomial approximations (1.1), (1.2), (1.3), and $\cosh x$ vs. $x$ for $x \in [-3, 3]$. 

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The solution $y_2$ with $y_2(0) = 0$, $y'_2(0) = 1$ is $\sinh x$, which we showed had a series solution given by

$$y_2 = \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$ 

The first three approximations to the series are then given by

$$y_2 \approx x, \quad (2.1)$$
$$y_2 \approx x + \frac{x^3}{6}, \quad (2.2)$$
$$y_2 \approx x + \frac{x^3}{6} + \frac{x^5}{120}. \quad (2.3)$$

These approximations are graphed below. Note that with each increasing term, the range of $x$ for which the polynomial is a good approximation widens.

In increasing order of thickness: Polynomial approximations $(2.1)$, $(2.2)$, $(2.3)$, and $\sinh x$ vs. $x$ for $x \in [-3, 3]$. 