Math 349 – Exam 1

Total time: 75 minutes
Total score: 100 points

Name:

There are 6 problems giving a total of 100 points. Please write your answers clearly. The use of notes and/or books is not permitted. Good luck!

1. (25 points) For which real numbers a does the following system have (i) no solution, (ii) a unique solution, (iii) infinitely many solutions.

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \,. \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right) \xrightarrow{5} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 7 & -14 & 10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right)$$

$$\xrightarrow{5} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 7 & -14 & 10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)$$

$a^2 - 16 = (a-4)(a+4)$

5 If $a = -4$, then we have $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 7 & -14 & 10 \\ 0 & 0 & 0 & -8 \end{array}\right)$. ($-8 \neq 0$)

The System is inconsitent.

5 If $a = 4$, then we have $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 7 & -14 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

The System has infinitely many solutions.

5 If $a \neq \pm 4$, then we have $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 7 & -14 & 10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)$. ($a^2-16 \neq 0$)

The System has a unique Solution.

2. (15 points) Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

Use the Gauss-Jordan method to find A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

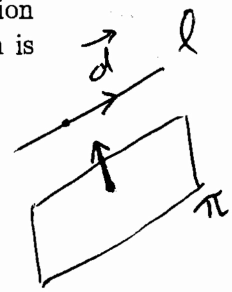
$$\overset{2}{\rightarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 3 & -1 & 0 \\ 0 & 3 & -4 & 2 & 0 & -1 \end{array}\right) \overset{2}{\rightarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 3 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & -1 \end{array}\right)$$

$$\overset{2}{\rightarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -5 & 3 & -2 \end{array}\right) \overset{2}{\rightarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -6 & 4 & -3 \\ 0 & 0 & 1 & -5 & 3 & -2 \end{array}\right)$$

$$\overset{2}{\rightarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -4 & 3 & -2 \\ 0 & 1 & 0 & -6 & 4 & -3 \\ 0 & 0 & 1 & -5 & 3 & -2 \end{array}\right) \overset{2}{\rightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -5 & 4 \\ 0 & 1 & 0 & -6 & 4 & -3 \\ 0 & 0 & 1 & -5 & 3 & -2 \end{array}\right)$$

(3) $A^{-1} = \begin{pmatrix} 8 & -5 & 4 \\ -6 & 4 & -3 \\ -5 & 3 & -2 \end{pmatrix}$

3. (15 points) (i) Determine the value(s) of a so that the line with parametric equation $x = -3 + t, y = 2 - t, z = 1 + at$ is parallel to the plane whose general equation is $3x - 5y + z = 3$.



A directional vector of the line is $\begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix}$

A normal vector of the plane is $\begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$.

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The line is parallel to the plane if & only if $\begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} = 0$ if & only if $a = -8$.

(ii) Find the distance from the point $Q = (3, 2, 5)$ to the plane whose general equation is $2x + 3y - z = 0$.

$\vec{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

Choose P to be any point on the plane, say

$P = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

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$\vec{PQ} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}$

$$\text{proj}_{\vec{n}}(\vec{PQ}) = \frac{\begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}}{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}} \vec{n} = \frac{8+3-4}{4+9+1} \vec{n}$$

$$= \frac{7}{14} \vec{n} = \frac{1}{2} \vec{n}$$

$$\|\text{proj}_{\vec{n}}(\vec{PQ})\| = \sqrt{\frac{4}{4} + \frac{9}{4} + \frac{1}{4}} = \sqrt{\frac{14}{4}} = \frac{\sqrt{14}}{2}$$

4. (15 points) Let

$$v_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Let $W = \text{span}\{v_1, v_2, v_3\}$. Determine if $W = \mathbf{R}^3$. Justify your answer.

Answer. $W \neq R^3$. 3

Reason:

$$\left(\begin{array}{ccc|c} 2 & -2 & 1 & a \\ 3 & -1 & 2 & b \\ 1 & -3 & 0 & c \end{array}\right) \overset{3}{\rightarrow} \left(\begin{array}{ccc|c} 1 & -3 & 0 & c \\ 0 & 4 & 1 & a-2c \\ 0 & 8 & 2 & b-3c \end{array}\right)$$

$$\overset{3}{\rightarrow} \left(\begin{array}{ccc|c} 1 & -3 & 0 & c \\ 0 & 4 & 1 & a-2c \\ 0 & 0 & 0 & b+c-2a \end{array}\right)$$

3

If $b+c-2a \neq 0$, then the system

$$\begin{pmatrix} 2 & -2 & 1 \\ 3 & -1 & 2 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$ has no solution.

So for example, 3 $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin \text{Span}\{v_1, v_2, v_3\}$.

5. (15 points) Let A be a 3×3 matrix which is NOT invertible. Determine which of the following statements are true, and which are false. Justify your answers.

(i) $AX = \mathbf{0}$ has a nontrivial solution (where $X = (x_1, x_2, x_3)^\top$ and $\mathbf{0} = (0, 0, 0)^\top$).

4 Yes. Since the columns of A are linearly dependent.

(ii) $AX = b$ has at least one solution for every b in $\mathbf{R}^3$.

4 NO. For example $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has no solutions.

(iii) There is a 3×3 matrix B such that AB is invertible.

4 NO. If $\exists B$ s.t. AB is invertible, then A is invertible.

(iv) There is a b in $\mathbf{R}^3$ such that $\text{rank}[A|b] > \text{rank}(A)$.

3 Yes. Since A is not invertible, rank(A) is at most 2. And the columns of A doesn't span $\mathbb{R}^3$. Choose $b \in \mathbb{R}^3$ which is not in the span of the columns of A. Then $\text{rank}[A|b] > \text{rank}(A)$.

6. (i) (8 points) If possible, express the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}$$

as a product of elementary matrices.

$$\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix} \xrightarrow{E_1} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \xrightarrow{E_2} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \xrightarrow{E_3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2 2 2

$$E_1 = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

So $E_3 E_2 E_1 A = I$.

Hence $A = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$

$$= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

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(ii) (7 points) Prove or Disprove: Let **u**, **v**, **w** be vectors in $\mathbf{R}^n$. If the sets $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}\}$, $\{\mathbf{v}, \mathbf{w}\}$ are each linearly independent, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

Let $u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$; $w = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. (5)

Then $\{u, v\}$ is L.I.

$\{v, w\}$ is L.I

& $\{u, w\}$ is L.I. But $\{u, v, w\}$ is L.D (2)

Since $w = u + v$.