Math 845, Midterm Exam

Fall 2009

1. (i) Let p be a prime, $k = p$ or $p+1$, and $P \in Syl_p(S_k)$. Prove that $|N_{S_k}(P)| = p(p-1)$.

(ii) Prove that there does not exist a simple group of order 396.

proof (i): If $k = p$ or $p+1$, then $P \in Syl_p(S_k)$ has order p. Therefore P is a cyclic group of order p, which is generated by a p-cycle in S_k.

$\therefore$ The number of Sylow p-subgroup

$$= \frac{\text{the \# of } p\text{-cycles in } S_k}{\text{the \# of } p\text{-cycles in } P}$$

$$= \frac{k(k-1)\cdots(k-p+1)\cancel{/p}}{(p-1)} = \frac{k(k-1)\cdots(k-p+1)}{p(p-1)}.$$

If $k = p$, the above number $= \dfrac{p(p-1)\cdots 1}{p(p-1)} = (p-2)!$

If $k = p+1$, then above number $= \dfrac{(p+1)p\cdots 2}{p(p-1)} = (p+1)\cdot(p-2)!$

In either case, we have $|N_{S_k}(P)| = \dfrac{k!}{\text{the \# of Sylow } p\text{-Subgps}} = p(p-1)$. This completes the proof of (i).

(ii) Assume to the contrary that G is a simple group of order $396 = 2^2 \cdot 3^2 \cdot 11$. Then $n_{11}(G) = 12$.

Let $P \in Syl_{11}(G)$. Then $[G : N_G(P)] = 12$; hence $|N_G(P)| = 33$.

Consider the action of G on the set of left cosets of $N_G(P)$ in G. Since G is simple, the action must be faithful. Hence there is an embedding $G \hookrightarrow S_{12}$. To simplify notation, we simply consider that $G \leq S_{12}$. $P \in Syl_{11}(G) \Rightarrow P \in Syl_{11}(S_{12})$. By part (i), we have $|N_{S_{12}}(P)| = 11 \cdot (11-1) = 110$. Since $N_G(P) \leq N_{S_{12}}(P)$, we would have $33 \mid 110$, which is clearly absurd. So we can't have a simple group of order 396.

2. Assume that G is a group, $x, y \in G$ and both x & y commute with $[x, y]$. Prove that for all positive integer n,

$$(xy)^n = x^n y^n [y, x]^{n(n-1)/2}.$$

Proof: We use induction on n.

$n=1$. There is nothing to prove.

First we note that if x & y commute with $[x,y]$, then x & y commute with $[y,x]$. (This is straightforward to prove. I omit the details).

Secondly, we claim that if x & y commute with $[y,x]$, then

$$[y, x]^n = y^{-n} x^{-1} y^n x, \text{ for all positive integers } n.$$

(i) If $n=1$, then there is nothing to prove

(ii) Assume that the conclusion is true for $(n-1)$.

(iii) $[y,x]^n = [y,x]\cdot[y,x]^{n-1}$

$$= y^{-1}x^{-1}yx \cdot y^{-(n-1)}x^{-1}y^{n-1}x$$

$$= y^{-(n-1)}(y^{-1}x^{-1}yx)x^{-1}y^{n-1}x$$

$$= y^{-n}x^{-1}y\cdot(xx^{-1})\,y^{n-1}x$$

$$= y^{-n}x^{-1}y^{n}x.$$

Now we return to the proof that $(xy)^n = x^n y^n [y,x]^{n(n-1)/2}$.

By what we just proved, we have

$$y^{-(n-1)}x^{-1}y^{n-1}x = [y,x]^{n-1}.$$

$$\Downarrow$$

$$y^{n-1}x = x[y,x]^{n-1}y^{n-1}$$

$$\Downarrow$$

$$y^{n-1}xy = x[y,x]^{n-1}y^{n} = xy^{n}[y,x]^{n-1}$$

$$\Downarrow$$

$$x^{n-1}y^{n-1}xy = x^{n}y^{n}[y,x]^{n-1}$$

$$\Downarrow$$

$$(*)\quad x^{n-1}y^{n-1}xy\,[y,x]^{\frac{(n-1)(n-2)}{2}} = x^{n}y^{n}[y,x]^{\frac{n(n-1)}{2}}.$$

Now $(xy)^n = (xy)^{n-1}xy$

$$\overset{\text{ind. hyp}}{=} x^{n-1}y^{n-1}[y,x]^{\frac{(n-1)(n-2)}{2}}xy \overset{\text{by}(*)}{=} x^{n}y^{n}[y,x]^{n(n-1)/2}$$

□

3. Classify groups of order $4p$, where p is a prime > 3.

Let G be a group of order $4p$, where p is a prime >3. Then $n_p(G) \equiv 1 \pmod{p}$ & $n_p(G) \mid 4 \Rightarrow n_p(G) = 1$. Hence the Sylow p-subgroup C_p of G is normal. Let Q be a Sylow 2-subgroup of G. Then

$$G = C_p \rtimes Q, \quad \text{where } |Q| = 4.$$

There are 4 isomorphism types of G when $p \equiv 3 \pmod 4$. These are

$G_1 = C_p \times C_4$, abelian

$G_2 = \langle a, b \mid a^p = 1 = b^4,\ bab^{-1} = a^{-1} \rangle$

$G_3 = C_p \times C_2 \times C_2$, abelian

$G_4 = \langle a, b, c \mid a^p = b^2 = c^2 = 1,\ bab^{-1} = a^{-1},\ ca = ac \rangle$.

There are 5 isomorphism types of G when $p \equiv 1 \pmod 4$. These are

$G_1 = C_p \times C_4$, abelian

$G_2 = \langle a, b \mid a^p = 1 = b^4,\ bab^{-1} = a^{-1} \rangle$

$G_3 = C_p \times C_2 \times C_2$, abelian

$G_4 = \langle a, b, c \mid a^p = b^2 = c^2 = 1,\ bab^{-1} = a^{-1},\ ca = ac \rangle$

$G_5 = \langle a, b \mid a^p = 1 = b^4,\ bab^{-1} = a^{i_0} \rangle$, where $i_0 \pmod p$ has order 4 in $U(\mathbb{Z}/p\mathbb{Z})$.

5. Let G be a permutation group on Ω, where $|\Omega| = n$. For $0 \le i \le n$, let P_i be the proportion of the elements of G which have exactly i fixed pts, and define $P_G(x) = \sum_{i=0}^{n} P_i x^i$. Also for $0 \le i \le n$, let F_i be the number of orbits of G acting on i-tuples of distinct elements of Ω, and let $F_G(x) = \sum_{i=0}^{n} F_i x^i / i!$. Prove that $F_G(x) = P_G(x+1)$.

Proof: We use the Orbit-Counting lemma to compute F_i. Note that An element of G with j fixed points has $j(j-1)\cdots(j-i+1)$ fixed i-tuples (of distinct elements of Ω), where $i \le j$. Hence

$$F_i = \sum_{j=i}^{n} P_j \, j(j-1)\cdots(j-i+1).$$

It follows that

$$F_G(x) = \sum_{i=0}^{n} F_i x^i / i!$$

$$= \sum_{i=0}^{n} x^i \sum_{j=i}^{n} P_j \binom{j}{i}$$

$$= \sum_{j=0}^{n} P_j \sum_{i=0}^{j} \binom{j}{i} x^i$$

$$= \sum_{j=0}^{n} P_j (1+x)^j.$$ Hence $F_G(x) = P_G(x+1)$.