Ch4. Markov Chain

Def: A stochastic process $\{X_t, t \in T\}$ is a collection of random variables X_t, $t \in T$. It is determined by
(i). State space S, i.e. the range of possible value of X_t.
(ii). index set T which can be discrete (finite or countable) or continuous.
(iii). dependency relations between variables.

Def: A stochastic process $\{X_n, n = 0, 1, 2 \cdots\}$ is a Markov Chain if $S = \{0, 1, 2, \cdots\}$ and

$$\mathbb{P}\left(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\right) = P_{ij}$$

for all $i_0, i_1, \cdots, i_{n-1}, i, j \in S$ and $n \geq 0$.

- $X_n = i$ means that the process is in the state $i \in S$ at the time n.

An Alternative Def: For any $1 < k < n$, conditional on the present $X_k = i_k$, the past $(X_1, \cdots, X_{k-1})$ and the future $(X_{k+1}, \cdots, X_n)$ are independent.

• By using the fact that $\mathbb{P}(\bullet|F)$ is a probability measure, we have

$$\begin{aligned}
& \mathbb{P}\left(X_{n+1}=j|X_n=i\right) \\
= & \sum_{i_{n-1},\cdots,i_1,i_0\in S} \mathbb{P}\left(X_{n+1}=j|X_n=i,X_{n-1}=i_{n-1},\cdots,X_0=i_0\right) \\
& \cdot\mathbb{P}\left(X_{n-1}=i_{n-1},\cdots,X_0=i_0|X_n=i\right) \\
= & P_{ij}\cdot\sum_{i_{n-1},\cdots,i_1,i_0\in S}\mathbb{P}\left(X_{n-1}=i_{n-1},\cdots,X_0=i_0|X_n=i\right) \\
= & P_{ij}
\end{aligned}$$

• $P_{ij}=\mathbb{P}\left(X_{n+1}=j|X_n=i\right)$ is called one-step transition probability from state i to state j.

• $P_{ij}\geq 0$, $\sum_{j\in S}P_{ij}=1$ which is easy to check.

Let

$$\mathbf{P}=\begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

be the matrix of the one-step transition prob. P_{ij}.

Ex: Rain or no rain, two states Markov Chain.

$$\mathbf{P} = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$$

Ex: A random walk model. $S = \{0, \pm 1, \pm 2, \cdots\}$
$P_{i,i+1} = p$, $P_{i,i-1} = 1 - p = q$

$$\mathbf{P} = \begin{pmatrix} \cdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & q & 0 & p & 0 & 0 & \cdots \\ \cdots & 0 & q & 0 & p & 0 & \cdots \\ \cdots & 0 & 0 & q & 0 & p & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \end{pmatrix}$$

Define the n-step transition probabilities

$$P_{ij}^{(n)} = \mathbb{P}(X_{n+l} = j | X_l = i)$$

(why it does not depend on l?). Then we have the Chapman-Kolmogorov equation

$$P_{ij}^{(n+m)} = \sum_{k \in S} P_{ik}^{(n)} \cdot P_{kj}^{(m)}.$$

Pf: What else? Conditioning!

In terms of matrix, we let $\mathbf{P}^{(n)} = \left(P_{ij}^{(n)}\right)$. Then

$$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)} \cdot \mathbf{P}^{(m)}, \quad \mathbf{P}^{(1)} = \mathbf{P}.$$

So $\mathbf{P}^{(n)} = \mathbf{P}^n$ =matrix multiplication.

Ex: Rain or no rain, two states Markov Chain.

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

Then

$$\mathbf{P}^4 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}.$$

So that $\mathbb{P}(\text{rain 4 days from today}|\text{raining today}) = P_{00}^{(4)} = 0.5749$.

• Initial distribution: $\alpha_i = \mathbb{P}(X_0 = i), i \geq 0$, $\sum_{i\geq 0} \alpha_i = 1$. Then

$$\mathbb{P}(X_n = j) \;=\; \sum_{i\geq 0} \mathbb{P}(X_n = j|X_0 = i) \cdot \mathbb{P}(X_0 = i) = \sum_{i\geq 0} P_{ij}^{(n)} \cdot \alpha_i$$

So if $\alpha_0 = 0.4$, $\alpha_1 = 0.6$ in the above example, then $\mathbb{P}(X_4 = 0) = 0.4 \cdot P_{00}^{(4)} + 0.6 P_{10}^{(4)} = 0.570$.

Examples of Markov Chain (M.C.)

Key; A sure way to check that a process is a Markov chain: Writing down its transition matrix!

Let $\xi, \xi_1, \cdots, \xi_n \cdots$ be i.i.d r.v's with $\mathbb{P}(\xi_i = k) = a_k \geq 0$, $k \geq 0$, $\sum_{k=0}^{\infty} a_k = 1$.

• Partial sum $S_n = \xi_1 + \cdots + \xi_n$, $n \geq 1$, $S_0 = 0$, is a M.C. with the transition matrix

$$\mathbf{P} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

• The maximum partial sum $Y_n = \max_{0 \leq i \leq n}(S_i - i)$ is NOT a M.C. in general.

• Successive maxima $M_n = \max(\xi_1, \xi_2, \cdots, \xi_n)$, $M_0 = 0$, is a M.C. with the transition matrix

$$\mathbf{P} = \begin{pmatrix} A_0 & a_1 & a_2 & \cdots \\ 0 & A_1 & a_2 & \cdots \\ 0 & 0 & A_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $A_i = a_0 + a_1 + \cdots + a_i$.

• Moving particle around a circle.

The Ehrenfest Urn Model

There is a total of r balls in two urns, A and B. At each step, pick one of the r balls at random and move it to the other urn. Ehrenfest used this to model the division of air molecules between two chambers (of equal size and shape) which are connected by a small hole. For an interesting account of this chain, see Kac (1947): Random walk and the theory of Brownian motion, *Amer. Math. Monthly*, **54**, 369–391.

Let X_n be the number of balls in urn A with $X_0 = l$, $0 \leq l \leq r$. Then $\{X_n, n = 0, 1 \cdots\}$ is a M.C. To see this, note that

$$X_{n+1} = \begin{cases} X_n - 1 & \text{with prob. } X_n/r \\ X_n + 1 & \text{with prob. } (r - X_n)/r \end{cases}$$

and thus we have

$$P_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) = \begin{cases} i/r & \text{if } j = i - 1 \\ (r-i)/r & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

with initial distribution $\mathbb{P}(X_0 = l) = 1$.

Enlarge State Space to Form a M.C.

Suppose that whether or not rains today depends on previous weather conditions through the last two days only. Let Y =yesterday, D=Today, T =Tomorrow, R =Rain, N =no-rain. Given

$$\mathbb{P}(T = R|D = R, Y = R) = 0.7$$
$$\mathbb{P}(T = R|D = R, Y = N) = 0.5$$
$$\mathbb{P}(T = R|D = N, Y = R) = 0.4$$
$$\mathbb{P}(T = R|D = N, Y = N) = 0.2$$

Note that for $Y_n =$ n's day condition with state space $S = \{0 = R, 1 = N\}$ is not a Markov chain (why not?).

Let X_n =($(n-1)'s$ day condition, n's day condition) with state space

$$S = \{0 = (R, R), 1 = (N, R), 2 = (R, N), 3 = (N, N)\}.$$

Then X_n is a Markov chain (why?) with

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}$$

Note that

$$\begin{aligned} \mathrm{P}_{00} &= \mathbb{P}(X_{n+1} = (R, R)|X_n = (R, R)) \\ &= \mathbb{P}(T = R|D = R, Y = R) = 0.7 \end{aligned}$$

Key ideas:

• Enlarge state space to form a M.C.

• Including time dimension to change time inhomogeneous M.C. to time homogeneous M.C.

A central limit theorem for the sock–sorting problem given by three different theories: Empirical process, Martingale and Markov Chain

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The paper appeared in *Progress in Probability*, Vol. **43**, 245-248, (1998).

Sock–sorting problem:

A collection of n different pairs of socks are scrambled in a laundry bag. Socks are drawn one at a time from the bag in random order, and laid on a table. When the mate of a sock on the table is drawn, the two are paired and put in a drawer. How much table space is required?

Some simple facts:

Let L_k be the number of socks on the table after k socks have been drawn. Then $\mathbb{E}\, L_k = \frac{k(2n-k)}{2n-1}$. Here is a way to see this. Let $X_{ij} = 1$ if the i-th and j-th socks drawn are a pair, 0 otherwise. Then the expected number of completed pairs after k draws is

$$\mathbb{E}\left(\sum_{1\le i<j\le k} X_{ij}\right) = \sum_{1\le i<j\le k} \mathbb{E}\, X_{ij} = \frac{k(k-1)}{2(2n-1)}$$

since $\mathbb{E}\, X_{ij} = 1/(2n-1)$ for all i and j.

We want to find a distributional limit for

$$\left\{\frac{L_k - \mathbb{E}\, L_k}{n^{1/2}}\right\}_{k=0}^{2n}$$

The problem can be solved in three ways, using

a) empirical process theory
b) diffusion limits of Markov chains
c) Martingale theory, integral operators

It is easy to see that

$$\mathbb{P}(L_{k+1} = j | L_k = i) = \begin{cases} \frac{2n-k-i}{2n-k} & \text{if} \quad j = i+1 \\ \frac{i}{2n-k} & \text{if} \quad j = i-1 \\ 0 & \qquad \text{otherwise} \end{cases}$$

Thus L_k is a time inhomogeneous M.C.

By including time as a part of state space (enlarge state space), $Y_k = (k, L_k)$ is a time homogeneous M.C. with the transition probability

$$P_{(i,k)(j,l)} = \begin{cases} \frac{2n-i-k}{2n-i} & \text{if} \quad j = i+1, l = k+1 \\ \frac{k}{2n-i} & \text{if} \quad j = i+1, l = k-1 \\ 0 & \qquad \text{otherwise} \end{cases}$$

for $0 \le i, j \le 2n$, $0 \le k, l \le n$.

What about transitions after $2n$ steps?

Stay at $(2n, 0)$ forever!

Bernoulli Process

Let $S = \{0, 1, \cdots\}$ and define the MC $\{Y_n, n \geq 0\}$ by $Y_0 = 0$ and

$$\mathbb{P}(Y_{n+1} = i + 1 | Y_n = i) = p, \quad \mathbb{P}(Y_{n+1} = i | Y_n = i) = 1 - p$$

with $0 < p < 1$, for all $n \geq 0$. One may think of Y_n as the number of heads in n tosses of a coin (hence as a sum of Bernoulli's). For fixed n, $Y_n \sim Bi(n, p)$. It is clear that

$$\mathbb{P}(Y_{m+n} = j | Y_m = i) = \binom{n}{j-i} p^{j-i} (1-p)^{n-j+i}, \quad 0 \leq j - i \leq n.$$

Viewed as a MC, Y_n is not a very interesting process. However, we can compact the state space into $S^M = \{0, 1, \cdots, M\}$ and define $X_n = Y_n$ modulo M. Is X_n a MC on S^M?

Markov's other chain (1910)

We follow the book of Grimmett and Stirzaker (2001). Let $Y_1, Y_3, Y_5, \cdots$ be a sequence of i.i.d r.v's such that

$$\mathbb{P}(Y_{2k+1} = -1) = \mathbb{P}(Y_{2k+1} = 1) = 1/2, \quad k = 0, 1, 2, \cdots,$$

and define $Y_{2k} - Y_{2k=1} Y_{2y+1}$, for $k - 1, 2, \cdots$. Then $Y_2, Y_4, \cdots$ is a sequence of i.i.d r.v's with the same distribution as Y_1. The entire sequence Y_n, $n \geq 1$ is pairwise independent and

$$p_{ij}(n) = \mathbb{P}(Y_{m+n} = j | Y_m = i) = 1/2 \quad \text{for all } n \text{ and } i, j = \pm 1$$

Thus the CK-equations are satisfied. Is Y_n, $n \geq 1$ a Markov Chain?

Classification of States

Def: State j is *accessible* from state i, denote by $i \to j$, if $P_{ij}^{(n)} > 0$ for some $n \geq 0$. Two state i and j *communicate* if $i \to j$ and $j \to i$, i.e $i \leftrightarrow j$.

Properties of the relation $i \leftrightarrow j$:

(i). $i \leftrightarrow i$ for all i.

(ii). if $i \leftrightarrow j$, then $j \leftrightarrow i$.

(iii). If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

Pf of (iii): Using the Chapman-Kolmogorov equation. Details in class.

- Two states i and j are in the same *class* iff $i \leftrightarrow j$.

- State space $S = \cup C_m$ where C_m are different (disjoint) classes.

Def: The M.C is *irreducible* if there is only one class.

Ex 4.9, 4.10

The first visit distribution:

$$f_{ij}^{(n)} = \mathbb{P}\left(X_1 \neq j, \cdots, X_{n-1} \neq j, X_n = j | X_0 = i\right) = \mathbb{P}\left(R_j = n | X_0 = i\right)$$

where $R_j = \min\{k \geq 1 : X_k = j\}$ is the first time to visit j.

- $f_{ij}^{(0)} = 0$, $f_{ij}^{(1)} = P_{ij}$.
- $P_{ij}^{(n)} = \sum_{k=0}^{\infty} f_{ij}^{(k)} \cdot P_{jj}^{(n-k)}$ for all $n \geq 1$.

The probability of ever return to i:

$$f_i = f_{ii} = \mathbb{P}(X_n = i \text{ for some } n \geq 1 | X_0 = i) = \mathbb{P}(R_i < \infty | X_0 = i)$$

Def: A state i is *recurrent* if $f_i = 1$ and *transient* if $f_i < 1$.

Prop. 4.1: A state i is recurrent (transient) iff

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \quad \left(\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty\right)$$

- $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \mathbb{E}\left(\sum_{n=1}^{\infty} 1_{(X_n = i)} | X_0 = i\right)$ which is the mean number of returns.
- Recurrent and transient are class property.

Ex: 4.11, 4.12: Determine the recurrent state.

Ex: 4.13 (Simple Symmetric Random Walk in $\mathbb{Z}^d$): The walk is at each transition equally likely to take one step to one of its $2d$ neighbors, i.e.

$$P_{(i_1,i_2,\cdots,i_d)(j_1,j_2,\cdots,j_d)} = \frac{1}{2d} \quad \text{if} \quad \sum_{k=1}^{d} |i_k - j_k| = 1.$$

Then the walk is recurrent for $d = 1, 2$ and transient for $d \geq 3$.

Def: The *period* of a state i is given by

$$d(i) = \text{g.c.d}\left\{n \geq 1 \,:\, P_{ii}^{(n)} > 0\right\}$$

where g.c.d means the great common divisor.

- The period is a class property, i.e. if $i \leftrightarrow j$, then $d(i) = d(j)$.

Simple Random Walk in $\mathbb{Z}^1$: The one step transition probability is $P_{i,i+1} = p$ and $P_{i,i-1} = q$ with $p + q = 1$. The period $d(i) = 2$ for all $i \in \mathbb{Z}$.

Def: A M.C is *aperiodic* if $d(i) = 1$ for all $i \in S$.

Def: A recurrent state i is *positive recurrent* if $m_i < \infty$, and *null recurrent* if $m_i = \infty$, where

$$m_i = \mathbb{E}\left(R_i | X_0 = i\right) = \sum_{n=1}^{\infty} n \cdot \mathbb{P}\left(R_i = n | X_0 = i\right) = \sum_{n=1}^{\infty} n \cdot f_{ii}^{(n)}$$

is the mean first return time.

- Both positive recurrent and null recurrent are class property.

- For a finite state M.C, all recurrent states are positive recurrent.

- After each visit of state i, the first return time $T_{ii} = \min\{k \geq 1 : X_k = i | X_0 = i\}$ back to state i is independent of the first return times after any of the other visits to the state (follows from the Markov property).

Def: Positive recurrent, aperiodic states are called *ergodic*.

Fundamental Theorem of Markov Chain

For an irreducible, ergodic (positive recurrent, aperiodic) Markov Chain,

$$\lim_{n\to\infty} P_{ij}^{(n)} = \lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_0 = i)$$

exists and is independent of i. Furthermore, letting

$$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)} = \lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_0 = i)$$

then π_j is the unique non-negative solution of

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j \geq 0, \quad \text{with} \quad \sum_{j=0}^{\infty} \pi_j = 1.$$

• Suppose for some $k \geq 1$, $P_{ij}^{(k)} > 0$ for all i, j in state space, then the MC is irreducible and ergodic, and hence π_j exists with $\pi_j > 0$.

• The equation $\pi_j = \sum_{i=0}^{\infty} P_{ij} \cdot \pi_i$ follows from $\mathbb{P}(X_{n+1} = j | X_0 = k) = \sum_{i=0}^{\infty} \mathbb{P}(X_{n+1} = j | X_n = i) \cdot \mathbb{P}(X_n = i) | X_0 = k)$ by taking $n \to \infty$. Similarly, $\sum_{j=0}^{\infty} P_{ij}^{(n)} = 1$ follows from $\sum_{j=0}^{\infty} \pi_j = 1$.

• The probability that the process is in state j after a long time: $\pi_j = \lim_{n\to\infty} \mathbb{P}(X_n = j)$.

• The long run mean proportion of time the process is in state j (or visit j):

$$\begin{aligned} \pi_j = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} P_{ij}^{(k)} &= \lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n}\sum_{k=0}^{n} 1_{\{X_k=j\}} | X_0 = i\right) \\ &= \lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n}\sum_{k=0}^{n} 1_{\{X_k=j\}}\right) \end{aligned}$$

This is the simplest illustration of the ergodic principle: Time averages equal space averages.

• π_j is the *stationary* distribution (probabilities), also called *equilibrium* distribution, since if $\mathbb{P}(X_k = j) = \pi_j$ for all $j \in S$, then for $n \geq k+1$ via induction

$$\mathbb{P}(X_n = j) = \sum_{i=0}^{\infty} \mathbb{P}(X_n = j | X_{n-1} = i) \cdot \mathbb{P}(X_{n-1} = i) = \sum_{i=0}^{\infty} P_{ij} \cdot \pi_i = \pi_j.$$

It is also clear that $\pi \mathbf{P}^n = \pi$.

•An irreducible MC has a stationary distribution iff MC is positive recurrent.

• $\pi_j = 1/m_j$ where m_j is the mean time of the first return. Note that it is easy to see $n\pi_i \cdot m_j \approx n$ for large n.

• In the matrix form,

$$\mathbf{P}^n \to \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \cdots \\ \pi_1 & \pi_2 & \pi_3 & \cdots \\ \pi_1 & \pi_2 & \pi_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and $\pi = \pi \cdot \mathbf{P}$ with $\pi = (\pi_1, \pi_2, \cdots)$. In fact, proof of the fundamental Thm of M.C is based on the fact that the maximum value of any given column of $\mathbf{P}^n$ is decreasing, the minimum value of any column of $\mathbf{P}^n$ is increasing and the difference is going to zero as $n \to \infty$.

•The equation $\pi = \pi\mathbf{P}$ is often called the (global) balance condition since we can write it as

$$\sum_i \pi_j P_{ji} = \sum_i \pi_i P_{ij}, \quad i.e. \quad \text{balance of proability flows.}$$

That is, for each state j, the probability of enter j is the same as the probability of leaving j.

• If the one step transition matrix $\mathbf{P}$ is a doubly stochastic matrix, i.e $\sum_{i=1}^{|S|} P_{ij} = 1$ for all $j \in S$, in addition to $\sum_{j=1}^{|S|} P_{ij} = 1$ for all $i \in S$, then $\pi_i = 1/|S|$ for $|S| < \infty$.

•In general, the balance (or equilibrium) equations are solved up to a constant factor. The factor is determined by the normalization condition $\sum_i \pi_i = 1$. For calculation by a computer, its often more convenient to use the solution

$$\pi = \mathbf{e} \cdot (\mathbf{P} + \mathbf{E} - \mathbf{I})^{-1}$$

where $\mathbf{e} = (1, 1, \cdots, 1)$ and $\mathbf{E} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$. Note that it can be shown that the inhomogeneous equation $\pi(\mathbf{P} + \mathbf{E} - \mathbf{I}) = \mathbf{e}$ has a unique solution which automatically satisfies both the balance and the normalization condition.

Ex: A model of class mobility.

Ex: Consider transition matrix $\mathbf{P} = \begin{bmatrix} 1-p & 1-p & p^2 \\ 1-p & 1-p & p^2 \\ 0 & 1-p & p \end{bmatrix}$ with $0 < p < 1$. Then $\pi = (\frac{(1-p)^2}{1-p(1-p)}, \frac{p(1-p)}{1-p(1-p)}, \frac{p^2}{1-p(1-p)})$.

King's Problem

The chess king, starting from the center cell in a 3×3 chessboard, moves freely and equally likely in each step. What is the approximate probability that the chess king is in the upper-right cell after one million steps?

	K	

1	2	3
4	0	5
6	7	8

1	2	1
2	0	2
1	2	1

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/5 & 2/5 & 2/5 \end{pmatrix}$$

Key ideas in solving King's Problem:

• Reduction on the number of states by suing symmetry. For the reduced M.C, $\pi_0 = 1/5, \pi_1 = 3/10, \pi_2 = 1/2$ by solving $\mathbf{\Pi} \cdot \mathbf{P} = \mathbf{\Pi}$, $\sum_{i=0}^{2} \pi_i = 1$.

• Stationary distribution can be seen via the proportion of out or in for a given cell, 3 for type 1, 5 for type 2 and 8 for type 0. Thus the total $4 \times 3 + 4 \times 5 + 1 \times 8 = 40$ and the probability in the upper-right corner is 3/40.

• Both the original and the reduced M.C are time reversible M.C. Hence the probability can be found easily by solving $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i, j \in S$ and $\sum_{i \in S} \pi_i = 1$.

Two by Two Case

Consider

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = V\Lambda V^{-1} = \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1-a-b \end{bmatrix} \cdot \frac{1}{a+b} \begin{bmatrix} b & a \\ 1 & -1 \end{bmatrix}$$

Thus

$$\begin{aligned} P^n = V\Lambda^n V^{-1} &= \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & (1-a-b)^n \end{bmatrix} \cdot \frac{1}{a+b} \begin{bmatrix} b & a \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{a+b} \begin{bmatrix} b + a(1-a-b)^n & a - a(1-a-b)^n \\ b - b(1-a-b)^n & a + b(1-a-b)^n \end{bmatrix} \\ &= \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b} \begin{bmatrix} a & -a \\ -b & b \end{bmatrix} \end{aligned}$$

and due to $|1-a-b| < 1$

$$P^n \to \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix}$$

Random Walk with Retaining Barrier

Consider a random walk on the non-negative integers with a retaining barrier at 0. The transition probabilities are

$$P_{00} = q, \quad P_{i,i+1} = p, \quad \text{for } i \geq 0, \quad P_{i,i-1} = q, \quad \text{for } i \geq 1,$$

where $p + q = 1$, $0 < p < 1$. Let $\rho = p/q$.

(a). If $\rho > 1$, then the MC is transient.

(b). If $\rho < 1$, then the MC is ergodic with stationary distribution $\pi_j = \rho^j(1 - \rho)$.

(c). $\rho = 1$, then the solution to $\pi = \pi P$ is $\pi_j = c$ and thus the MC is null recurrent.

Ex: "Up" and "Down" in a Production.

$$\begin{aligned}\text{Rate enter } j \text{ from } i &= \lim_{n\to\infty}\mathbb{P}(X_{n+1}=j, X_n=i)\\ &= \pi_i P_{ij}\end{aligned}$$

and

$$\begin{aligned}&\text{Rate of breakdown}\\ =\;& \lim_{n\to\infty}\mathbb{P}(X_{n+1}\in A^c, X_n\in A)\\ =\;& \lim_{n\to\infty}\sum_{i\in A}\sum_{j\in A^c}\mathbb{P}(X_{n+1}=j, X_n=i)\\ =\;& \sum_{i\in A}\sum_{j\in A^c}\pi_i P_{ij}\end{aligned}$$

- Reduction of a M.C to only two states, "Up" and "Down".

Ex: The Hardy-Weinberg Law and a M.C. in Genetics.

Ex: A Model for Algorithmic Efficiency.

Ex: A Probabilistic Algorithmic for the Satisfiability.

Ex: The Keg Index and a Mathematical Theory of Drunkenness. Ref: *Mathematical Intelligencer*, **24**, No 3, 34-36, (2002).