

## Solution of EXAM II, Spring 2008

1. (10 pts). Three individuals,  $A$ ,  $B$  and  $C$ , all require kidney transplants. If she does not receive a new kidney, then  $A$  will die after an exponential time with rate  $\mu_A$ , and  $B$  after an exponential time with rate  $\mu_B$ ,  $C$  after an exponential time with rate  $\mu_C$ . New kidney arrive in accordance with a Poisson process having rate  $\lambda$ . It has been decided that  $A$  is the first in line to receive a kidney (if  $A$  is still alive) and  $B$  is the second and  $C$  is the third. What is the probability that  $C$  obtains a new kidney?

**Sol 1:** We have  $P = P_1 + P_2 + P_3$  where

$$P_i = \mathbb{P}(C \text{ receives the } i\text{-th kidney}), \quad i = 1, 2, 3.$$

Then

$$\begin{aligned}
 P_1 &= \frac{\mu_A}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\mu_B}{\mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda} \\
 &\quad + \frac{\mu_B}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\mu_A}{\mu_A + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda} \\
 P_2 &= \frac{\mu_A}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda} \\
 &\quad + \frac{\lambda}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\mu_B}{\mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda} \\
 &\quad + \frac{\mu_B}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_A + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda} \\
 P_3 &= \frac{\lambda}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda}
 \end{aligned}$$

**Sol 2:** We have  $P = P_N + P_A + P_B + P_{AB}$  where

$$P_N = \mathbb{P}(\text{No one dies}) = P_3$$

$$P_A = \mathbb{P}(\text{A dies only})$$

$$= \frac{\mu_A}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda}$$

$$P_B = \mathbb{P}(\text{B dies only})$$

$$= \frac{\lambda}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\mu_B}{\mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda}$$

$$+ \frac{\mu_B}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_A + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda}$$

$$P_{AB} = \mathbb{P}(\text{Only A and B die}) = P_1$$

**Sol 3:** Starting with  $A, B, C$  on the waiting list, the list changes into  $B, C$  or  $A, C$ , and then into  $C$ . Anyone is off the list if he/she dies or receives a kidney. Thus the probability is

$$\frac{\lambda + \mu_A}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\lambda + \mu_B}{\mu_B + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda}$$

$$+ \frac{\mu_B}{\mu_A + \mu_B + \mu_C + \lambda} \cdot \frac{\lambda + \mu_A}{\mu_A + \mu_C + \lambda} \cdot \frac{\lambda}{\mu_C + \lambda}$$

2. (24 pts). Let  $S_1, S_2, \dots$  be the waiting time in a Poisson process  $\{N(t), t \geq 0\}$  of rate  $\lambda$ . Determine the following:

(a).  $\mathbb{P}(N(t) = 0)$ . **Ans:**  $e^{-\lambda t}$ .

(b).  $\mathbb{P}(N(t) = 0, N(2t) = 1)$ . **Ans:**  $\lambda t e^{-2\lambda t}$ .

(c).  $\mathbb{P}(N(t) = 2 | N(t+1) = 2)$ . **Ans:**  $(t/(t+1))^2$ .

(d).  $\mathbb{P}(N(t+1) = 4 | N(t) = 2)$ . **Ans:**  $\lambda^2 e^{-\lambda}/2$ .

(e).  $\mathbb{E}(N(t+1) | N(t) = 2)$ . **Ans:**  $2 + \lambda$ .

(f).  $\mathbb{P}(S_1 > t | N(2t) = 3)$ . **Ans:**  $1/8$ .

(g).  $\mathbb{E}(S_3 | N(t) = 1)$ . **Ans:**  $t + 2/\lambda$ .

(h).  $\mathbb{E}(S_2 | S_4 = 5)$ . **Ans:**  $5/2$ .

(i).  $\mathbb{E}(S_1 + S_2 + S_3)$ . **Ans:**  $6/\lambda$ .

(j).  $\mathbb{P}(S_{N(t)+1} > x | S_1 = t + 2x)$  for  $x > 0$ . **Ans:**  $1$ .

(k).  $\mathbb{P}(S_{N(t)+1} > 2t)$ . **Ans:**  $e^{-\lambda t}$ .

(l).  $\mathbb{E}(S_1 S_2 S_3 | N(t) = 3)$ . **Ans:**  $t^3/8$ .

3. (16 pts). A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of two per hour. The successive service times are independent exponential random variables with rate two per hour if there is only one customer in the shop, and with rate three per hour if there are two customers in the shop. What is the proportion of potential customers that enter the shop?

**Sol:** This is a birth/death process with

$$\lambda_0 = 2, \mu_0 = 0; \quad \lambda_1 = 2, \mu_1 = 2; \quad \lambda_2 = 0, \mu_2 = 3.$$

We can find by using the local (reversible) balance equation,

$$P_0 = 3/8, \quad P_1 = 3/8, \quad P_2 = 1/4.$$

Thus the proportion of potential customers that enter the shop is  $P_0 + P_1 = 3/4$ .

## Brownian Motion

Brownian motion (BM) is a process of tremendous practical and theoretical significance. Intuitively, Brownian motion corresponds to the concept of a homogeneous, continuous time, continuous random walk. One way to visualize Brownian paths is to consider a simple random walk on the real line, in which the walker starts at 0 and moves up or down by an amount  $\sqrt{dt}$  after each timeinterval of duration  $dt$ . To be more precise, the continuous path  $Z_n(t)$ , builded from linear interpolation of simple random walks, “converges” to Brownian motion  $B(t)$  where

$$Z_n(t) = \frac{1}{\sqrt{n}} \left( S_{[nt]} + (nt - [nt])X_{[nt]+1} \right)$$

and  $S_0 = 0$ ,  $S_k = \sum_{i=1}^k X_i$ ,  $X_i$  are i.i.d with  $\mathbb{P}(X_i = \pm 1) = 1/2$ .

- The existence of BM was first rigorously proved by Wiener (1923). So the BM is also called the Wiener process.

**Def:** A continuous time stochastic process

$\{B_t, 0 \leq t < \infty\}$  on  $\mathbb{R}$  is called a standard Brownian Motion (BM) if it has the following properties:

(i).  $B_0 = 0$ .

(ii). The increments of  $B_t$  are independent; i.e. for any finite set of times  $t_0 < t_1 < \dots < t_n$ , the r.v.'s  $B_{t_0}, B_{t_1} - B_{t_0}, \dots, B_{t_n} - B_{t_{n-1}}$  are ind.

(iii). For any  $0 \leq s \leq t$ , the increment  $B_t - B_s$  has normal distribution with mean 0 and variance  $t - s$ .

(iv). With probability one, the paths of  $B_t$  are continuous.

**Def:** A  $n$ -dimensional Brownian motion started at  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  is given by

$$B_t^x = (x_1 + B_t^{(1)}, \dots, x_n + B_t^{(n)})$$

where  $\{B_t^{(j)}\}$ ,  $1 \leq j \leq n$  are ind. standard BMs in  $\mathbb{R}$ .

## Basic Properties of Brownian Motion

- Symmetry: The process  $\tilde{B}_t = -B_t, t \geq 0$  is a BM.
- Scaling: For any  $c > 0$ , the process  $\tilde{B}_t = cB_{t/c^2}, t \geq 0$  is a BM.
- Time homogeneity: For any fixed  $s > 0$ , the process  $\tilde{B}_t = B_{t+s} - B_s, t \geq 0$  is a BM independent of  $\sigma(B_u, u \leq s)$ .
- Time inversion: The process

$$\tilde{B}_t = tB_{1/t}, \quad t > 0, \text{ and } \tilde{B}_0 = 0,$$

is a BM

- Brownian paths are Hölder continuous with exponent  $\gamma$  for any  $\gamma < 1/2$ .
- With probability one, Brownian paths are not Lipschitz continuous (and hence not differentiable) at any point.
- The total variation of BM is infinite a.s.

## Ways to Define an Integral

Consider a partition  $S = t_0 < t_1 < \dots < t_k = T$  of  $[S, T]$  with  $\Delta_j = t_{j+1} - t_j$  and  $\Delta = \max \Delta_k$ .

- Riemann Integral: For integrable function  $f(t)$  on interval  $[S, T]$ ,

$$\int_S^T f(t)dt = \lim_{\Delta \rightarrow 0} \sum_j f(t_j^*)(t_{j+1} - t_j)$$

for any  $t_j^* \in [t_j, t_{j+1}]$ .

- Riemann-Stieltjes Integral:

$$\int_S^T f(t)dG(t) = \lim_{\Delta \rightarrow 0} \sum_j f(t_j^*)(G(t_{j+1}) - G(t_j))$$

for any  $t_j^* \in [t_j, t_{j+1}]$ .

- Lebesgue Integral: Partition the range of Lebesgue integrable function.

- Wiener Integral: For square integrable function  $f(t)$  on  $[S, T]$ ,

$$\begin{aligned} \int_S^T f(t)dB_t(\omega) &= \lim_{n \rightarrow \infty} \int_S^T f_n(t)dB_t(\omega) \\ \int_S^T f_n(t)dB_t(\omega) &= f(t)B_t(\omega) \Big|_S^T - \int_S^T B_t(\omega)df_n(t) \end{aligned}$$

for bounded variation functions  $f_n$  and  $f_n \rightarrow f$ .

# Stochastic Integration

- The basic idea to define any integral is to start with a simple class of functions (such as step functions) and then extend it by some approximation procedures (such as integration by parts, isometry).
- For a random function  $f(t, \omega)$ , there are difficulties to define

$$\int_0^T \phi(t, \omega) dB_t(\omega) = \sum_j f(t_j^*, \omega) (B_{t_{j+1}}(\omega) - B_{t_j}(\omega))$$

for step function  $\phi(t, \omega) = \sum_j f(t_j^*, \omega) \cdot \chi_{[t_j, t_{j+1})}(t)$  where the points  $t_j^*$  belong to the intervals  $[t_j, t_{j+1}]$ .

- Ito Integral  $\int_S^T f(t, \omega) dB_t(\omega)$ : Take  $t_j^* = t_j$ , the left end point, and the approximation is in  $L^2$ .
- Stratonovich Integral  $\int_S^T f(t, \omega) \circ db_t(\omega)$ : Take  $t_j^* = (t_{j+1} + t_j)/2$ , the mid point, and the approximation is in  $L^2$ .
- The important thing to know about normal differential calculus is that it is the chain rule, the Fundamental Theorem of Calculus and Taylor's formula that enable us to calculate with functions. For stochastic differentials, Ito's formula will perform similar rules and formulas on random functions for us.