Exercise 6.4: Potential customers arrive at a single-server station in accordance with a Poisson process with rate λ. However, if the arrival finds n customers already in the station, then he will enter the system with probability α_n. Otherwise, he will leave. Assuming an exponential service rate μ, set this up as a birth and death process and determine the birth and death rates.

Exercise 6.6: Consider a birth and death process with birth rates $\lambda_i = (i+1)\lambda$, $i \geq 0$, and death rates $\mu_i = i\mu$, $i \geq 0$.
(a) Determine the expected time to go from state 0 to state 4.
(b) Determine the expected time to go from state 2 to state 5.
(c) Determine the variance of the time in (a).

Exercise 6.12: Each individual in a biological population is assumed to give birth at an exponential rate λ, and to die at an exponential rate μ. In addition, there is an exponential rate of increase θ due to immigration. However, immigration is not allowed when the population size is N or larger.
(a) Set this up as a birth and death model.
(b) If $N = 3$, $\lambda = \theta = 1$, and $\mu = 2$, determine the proportion of time that immigration is restricted.

Exercise 6.18: After being repaired, a machine functions for an exponential time with rate λ and then fails. Upon failure, a repair process begins. The repair process proceeds sequentially through k distinct phases. First a phase 1 repair must be performed, then a phase 2, and so on. The times to complete these phases are independent, with phase i taking an exponential amount of time with rate μ_i, $i = 1, 2, \cdots, k$.
(a) What proportion of time is the machine undergoing a phase i repair?
(b) What proportion of time is the machine working?

An Epidemic Model

A village with n people suffers an epidemic. Let X_t be the number of sick people at time t, $t \geq 0$, and suppose that $X_0 = 1$. If we model X_t as a continuous time Markov Chain, then a plausible model is to take the infinitesimal transition probability $\lambda_{i,i+1} = \lambda i \cdot (n+1-i)$ to be proportional to the number of encounters between sick and well people. All other $\lambda_{ij} = 0$. Now let T be the time at which the last member of the village succumbs to the disease. Since the waiting time to move from state i to state $i+1$ is exponential with intensity $\lambda_{i,i+1}$, we see that $\mathbb{E}(T) \approx 2(\ln n + \gamma)/(\lambda n)$, where $\gamma \approx 0.5772$ is Euler's constant. It is interesting that $\mathbb{E}(T)$ decreases with n for large n.

Renewal Theory

A counting process $\{N(t), t \geq 0\}$ is a renewal process if the interarrival time X_n (between $(n-1)$-th events and n-th events) are i.i.d with c.d.f $F(x) = \mathbb{P}(X_n \leq x)$ and mean $\mu = \mathbb{E}\, X_n > 0$. Let $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$, the time of the n-th renewal. Then

$$N(t) = \max\{n : S_n \leq t\}, \quad t \geq 0$$

and the distribution for fixed t is

$$\begin{aligned}
\mathbb{P}(N(t) = n) &= \mathbb{P}(N(t) \geq n) - \mathbb{P}(N(t) \geq n+1) \\
&= \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t) = F_n(t) - F_{n+1}(t)
\end{aligned}$$

where $F_n(t) = \mathbb{P}(S_n \leq t)$ and $f_n(t) = F_n'(t)$ is the n-fold convolution of $f = F'$.

•The **renewal function:** $m(t) = \mathbb{E}\, N(t) = \sum_{n=1}^{\infty} F_n(t)$ satisfies the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x) f(x) dx$$

if $F(t)$ has a density $f(t)$.

•$m(t)$ is linear iff $N(t)$ is a Poisson process.

Limit Theorems for Renewal Processes

SLLN: With probability one,

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty.$$

Elementary renewal Theorem:

$$\frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty.$$

CLT: Let $\mu = \mathbb{E}\, X$ and $\sigma^2 = \mathsf{Var}(X)$. Then

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N(t) - (t/\mu)}{\sqrt{t\sigma^2/\mu^3}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds.$$

Note that $\mathbb{E}\, N(t) \sim t/\mu$ and $\mathsf{Var}(N(t)) \sim t\sigma^2/\mu^3$ as $t \to \infty$. Equivalently,

$$\frac{N(t) - \mathbb{E}\, N(t)}{\sqrt{\mathsf{Var}(N(t))}} \Rightarrow N(0,1).$$

Renewal Equations

•**Fundamental Renewal Equations:**

$$g_0(t) = 1 + \int_0^t g_0(t-x)f(x)dx, \quad f(x) \geq 0.$$

•**General Renewal Equations:**

$$g(t) = h(t) + \int_0^t g(t-x)f(x)dx, \quad f(x) \geq 0, \int_0^\infty xf(x)dx < \infty$$

and $h(t)$ is bounded.

•**Connections (Analog of ODEs:**

$$g(t) = \int_0^t h(t-s)g_0'(s)ds.$$

•What is $\mathbb{E}\, S_{N(t)+1}$?

The Inspection Paradox

•What is the distribution of $X_{N(t)+1}$?

•For Poisson process with parameter λ, the age of the process at the time t, $A(t) = t - S_{N(t)}$ is independent of the excess life of the process at time t, $Y(t) = S_{N(t)+1} - t$. In addition, $Y(t) \sim$ expo(λ) and $A(t)$ is approximate expo(λ) for large t. Thus for t large, $\mathbb{E}\, X_{N(t)+1} = \mathbb{E}\, A(t) + \frac{1}{\lambda} \approx \frac{2}{\lambda}$

Ex: Consider a renewal process $\{N(t), t \geq 0\}$ having a gamma(r, λ) interarrival distribution. Then $\mathbb{P}(N(t) \geq n) = \sum_{i=nr}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$ and $m(t) = \sum_{i=r}^{\infty} \left[\frac{i}{r}\right] \frac{e^{-\lambda t}(\lambda t)^i}{i!}$ where $[i/r]$ is the greatest integer less than or equal to i/r.

Project V: Properties of Geometric R.V's

By comparing with known facts about exponential random variables, develop analog properties for geometric random variables as outlined in class. Detailed proofs and relevant references are helpful.

Project VI: Probability Computations

Let $N(t)$ be a Poisson process with rate $\lambda > 0$. Define the process (can you find the name from internet?)

$$X(t) = \int_0^t (-1)^{N(s)} ds.$$

For fixed $a > 0$ and $T > 0$, compute the probability that

$$\mathbb{P}\left(\sup_{0 \le t \le T} X(t) \le a \right)$$

and

$$\mathbb{P}\left(\sup_{0 \le t \le T} |X(t)| \le a \right).$$

Project VII: The Kidney Transplant Problem

There are n individuals, $A_1, \cdots, A_n$, all require kidney transplants. If A_i does not receive a new kidney, then A_i will die after an exponential time with rate μ_i. New kidney arrive in accordance with a Poisson process having rate λ. It has been decided that A_i is the i-th in line to receive a kidney (if A_i is still alive).

(i). What is the probability that A_n obtains a new kidney when $\mu_i = \mu$ for $1 \leq i \leq n$.

(ii). What is the probability that A_n obtains a new kidney for general μ_i, $1 \leq i \leq n$.

(iii). Construct a continuous time M.C to model the number of individuals who need kidney transplants when new kidney patients arrive in accordance with a Poisson process having rate ν.

Project VII: Barbershop

A small barbershop has two separated service chairs, one operated by father and the other by son. Potential customers arrive at a Poisson rate of λ. Among potential customers, a proportion of p_s requires service only from the son, a proportion of p_f requires service only from the father, and the remaining proportion of $p_b = 1 - p_s - p_f$ can be served by both son and father (when it is possible, a customer is equally likely to choose either the father or the son for service). The successive service times are independent exponential random variables with rate μ_s by the son and with rate μ_f by the father. Both father and son would like to know the proportion of potential customers (all three types) that enter the shop. Set up a continuous Markov Chain to model the problem under the assumption that there is/are (a) no additional Chair, (b) only one additional Chair, (c) exactly two additional Chairs for customers to wait. Make sure in each situation to provide state space used, holding rate at each state and transition probabilities between states.

What is the proportion of potential customers that enter the shop?