Applications and Extensions

- Sampling a Poisson process
- An Infinite Server Queue
- Minimizing the Number of Encounters
- Tracking the Number of SARS Infections
- Nonhomogeneous Poisson Process
- The Output Process of the M/G/∞ Queue
- Compound Poisson Process
- Busy Periods in Single-Server Poisson Arrival Queues
- The Coupon Collecting Problem

Second Proof of Equivalence of various definitions

Pf 2: Generating function (Laplace transform) approach. Consider $g(t,u) = \mathbb{E}\exp(-uN(t))$, $t,u \geq 0$. Then a differential equation in t can be obtained by finding infinitesimal changes:

$$\begin{aligned} g_u(t+h) &= g(t+h,u) = \mathbb{E}\exp(-uN(t+h)) \\ &= \mathbb{E}\exp(-uN(t)) \cdot \mathbb{E}\exp(-u(N(t+h)-N(t))) \\ &= g_u(t) \cdot \mathbb{E}\exp(-uN(h)) \end{aligned}$$

for $h > 0$ and

$$\begin{aligned} \mathbb{E}\exp(-uN(h)) &= \mathbb{P}(N(h)=0) + e^{-u}\mathbb{P}(N(h)=1) + o(h) \\ &= 1 + \lambda(e^{-u}-1)h + o(h). \end{aligned}$$

Similar result can be obtained for $h < 0$, $t+h > 0$ and we obtain

$$g_u'(t) = \lim_{h\to 0} \frac{g_u(t+h) - g(t)}{h} = g_u(t)\lambda(e^{-u}-1).$$

Thus $g(t,u) = \exp(\lambda t(e^{-u}-1))$ which is the generating function in u of a Poisson r.v. with mean λt.

•Note the roles of u and t.

Thinning of a Poisson Process

Suppose that $N(t)$ is a Poisson process with rate λ and that each event can be classified as type i with probability p_i, $\sum p_i = 1$ independent of all other events.

Let $N_i(t)$ be the number of type i events that have occurred up to time t. Then $N_i(t)$, $i = 1, \cdots m$ are independent Poisson processes having rates $p_i\lambda$, respectively.

Outline of Proof: First compute that joint distribution function of $(N_1(t), \cdots, N_m(t))$ by conditioning on $N(t)$. Then show $N_i(t)$ are processes with independent increments.

•Superposition of Poisson processes Let $N_i(t)$ be independent Poisson processes with rates λ_i, respectively. Then $N(t) := \sum N_i(t)$ is a Poisson processes with rate $\lambda = \sum \lambda_i$.

•Does a Poisson process with rate $\lambda > 0$ have a stationary distribution? If so, find it. If not, explain why it does not.

•A hitchhiker approaches a major interstate at a certain place and time of day where vehicles are known to pass according to a Poisson process with a rate of 200 cars per hour.

(a) If each car here will pick up the hitchhiker with probability 0.05, what is the hitchhiker's expected waiting time?

(b) What is the expected number of cars that will pass the hitchhiker before he/she is picked up?

(c) Now suppose that 1 in 100 cars that pass this point are highway patrol cars and if each patrol car will stop to question/warn/ticket the hitchhiker with probability 0.7, what is the probability that the hitchhiker will get a ride before being bothered by the highway patrol?

Exer. 5.44 Cars pass a certain street location according to a Poisson process with rate λ. A woman who wants to cross the street at that location waits until she can see that no cars will come by in the next T time units.
(a) Find the probability that her waiting time is zero. **Ans:** $e^{-\lambda T}$.
(b) Find her expected waiting time.
Hint: Condition on the time of the first car.
Sol: Let W denote the waiting time and let S denote the time until the first car. Then

$$\begin{aligned} \mathbb{E}\,W &= \int_0^{\infty} \mathbb{E}\,(W|S=t] \cdot \lambda e^{-\lambda t} dt \\ &= \int_0^{T} \mathbb{E}\,(W|S=t) \cdot \lambda e^{-\lambda t} dt + \int_T^{\infty} \mathbb{E}\,(W|S=t) \cdot \lambda e^{-\lambda t} dt \\ &= \int_0^{T} (t + \mathbb{E}\,W) \cdot \lambda e^{-\lambda t} dt + ?? \end{aligned}$$

Hence, $\mathbb{E}\,W = ??$

Exer. 5.49 Events occur according to a Poisson process with rate λ. Each time an event occurs, we must decide whether or not to stop, with our objective being to stop at the last event to occur prior to some specified time T, where $T > 1/\lambda$. That is, if an event occurs at time t, $0 \le t \le T$, and we decide to stop, then we win if there are no additional events by time T, and we lose otherwise. If we do not stop when an event occurs and no additional events occur by time T, then we lose. Also, if no events occur by time T, then we lose. Consider the strategy that stops at the first event to occur after some fixed time s, $0 \le s \le T$.

(a) Using this strategy, what is the probability of winning?
(b) What value of s maximizes the probability of winning?
(c) Show that one's probability of winning when using this strategy with the value of s you specified in part (b) is $1/e$.

EXAM II, Spring 2008

1. (10 pts). Three individuals, A, B and C, all require kidney transplants. If she does not receive a new kidney, then A will die after an exponential time with rate μ_A, and B after an exponential time with rate μ_B, C after an exponential time with rate μ_C. New kidney arrive in accordance with a Poisson process having rate λ. It has been decided that A is the first in line to receive a kidney (if A is still alive) and B is the second and C is the third. What is the probability that C obtains a new kidney?

2. (24 pts). Let S_1, S_2, $\cdots$ be the waiting time in a Poisson process $\{N(t), t \geq 0\}$ of rate λ. Determine the following:

(a). $\mathbb{P}(N(t) = 0)$.

(b). $\mathbb{P}(N(t) = 0,\, N(2t) = 1)$.

(c). $\mathbb{P}(N(t) = 2 \,|\, N(t+1) = 2)$.

(d). $\mathbb{P}(N(t+1) = 4 \,|\, N(t) = 2)$.

(e). $\mathbb{E}\,(N(t+1) \,|\, N(t) = 2)$.

(f). $\mathbb{P}(S_1 > t \,|\, N(2t) = 3)$.

(g). $\mathbb{E}\,(S_3 \,|\, N(t) = 1)$.

(h). $\mathbb{E}\,(S_2 \,|\, S_4 = 5)$.
(i). $\mathbb{E}\,(S_1 + S_2 + S_3)$.
(j). $\mathbb{P}(S_{N(t)+1} > x \,|\, S_1 = t + 2x)$ for $x > 0$.
(k). $\mathbb{P}(S_{N(t)+1} > 2t)$.
(l). $\mathbb{E}\,(S_1 S_2 S_3 | N(t) = 3)$.

3. (16 pts). A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of two per hour. The successive service times are independent exponential random variables with rate two per hour if there is only one customer in the shop, and with rate three per hour if there are two customers in the shop. What is the proportion of potential customers that enter the shop?

Continuous-Time Markov Chains

Def: A continuous-time stochastic process $\{X(t), t \geq 0\}$, taking on values in the set of nonnegative integers, is a *continuous-time Markov chain* if for all $s, t \geq 0$ and nonnegative integers $i, j, x(u)$, $0 \leq u < s$,

$$\begin{aligned} \mathbb{P}\{X(t+s) &= j | X(s) = i, X(u) = x(u), 0 \leq u < s\} \\ &= \mathbb{P}\{X(t+s) = j | X(s) = i\} \end{aligned}$$

In other words, a continuous-time Markov chain is a stochastic process having the Markovian property that the conditional distribution of the future $X(t+s)$ given the present $X(s)$ and the past $X(u), 0 \leq u < s$, depends only on the present and is independent of the past. If, in addition,

$$P_{ij}(t) = \mathbb{P}\{X(t+s) = j | X(s) = i\}$$

is independent of s, then the continuous-time Markov chain is said to have *stationary* or *homogeneous* transition probabilities.

- Compare with discrete time M.C.

HW: Ch 6: 9th edition [8th](7th) . **2***, **4***, **13**, **14**, **19***, **24***, **30**, **31**, If we let T_i denote the amount of time that the process stays in state i before making a transition into a different state, then

$$\mathbb{P}\{T_i > s + t | T_i > s\} = \mathbb{P}\{T_i > t\}$$

for all $s, t \geq 0$. Hence, the random variable T_i is *memoryless* and must thus be *exponentially* distributed.

In fact, the above gives us another way of defining a continuous-time Markov chain. Namely, it is a stochastic process having the properties that each time it enters state i

(i). the amount of time it spends in that state before making a transition into a different state is exponentially distributed with mean, say, $1/v_i$, and

(ii). when the process leaves state i, it next enters state j with some probability, say, P_{ij}. Of course, the P_{ij} must satisfy

$$P_{ii} = 0, \quad \text{all} \quad i$$
$$\sum_j P_{ij} = 1, \quad \text{all} \quad i$$

In other words, a continuous-time Markov chain is a stochastic process that moves from state to state in accordance with a (discrete-time) Markov chain, but is such that the amount of time it spends in each state, before proceeding to the next state, is exponentially distributed. In addition, the amount of time the process spends in state i, and the next state visited, must be independent random variables. For if the next state visited were dependent on T_i, then information as to how long the process has already been in state i would be relevant to the prediction of the next state–and this contradicts the Markovian assumption.

Ex. (A Shoeshine Shop): Consider a shoeshine establishment consisting of two chairs–chair 1 and chair 2. A customer upon arrival goes initially to chair 1 where his shoes are cleaned and polish is applied. After this is done the customer moves on to chair 2 where the polish is buffed. The service times at the two chairs are assumed to be independent random variables which are exponentially distributed with respective rates μ_1 and μ_2. Suppose that potential customers arrive in accordance with a Poisson process having rate λ, and that a potential customer will only enter the system if both chairs are empty.

The preceding model can be analyzed as a continuous-time Markov chain, but first we must decide upon an appropriate state space. Since a potential customer will enter the system only if there are no other customers present, it follows that there will always either be 0 or 1 customer in the system. However, if there is 1 customer in the system, then we would also need to know which chair he was presently in. Hence, an appropriate state space might consist of the three states 0, 1, and 2 where the states have the following interpretation:

State	*Interpretation*
0	system is empty
1	a customer is in chair 1
2	a customer is in chair 2

It is easy to verify that

$$v_0 = \lambda, \quad v_1 = \mu_1, \quad v_2 = \mu_2,$$
$$P_{01} = P_{12} = P_{20} = 1.$$

Birth and Death Processes

Consider a system whose state at any time is represented by the number of people in the system at that time. Suppose that whenever there are n people in the system, then
(i) new arrivals enter the system at an exponential rate λ_n, and
(ii) people leave the system at an exponential rate μ_n.
That is, whenever there are n persons in the system, then the time until the next arrival is exponentially distributed with mean $1/\lambda_n$ and is independent of the time until the next departure which is itself exponentially distributed with mean $1/\mu_n$. Such a system is called a birth and death process. The parameters $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ are called respectively the arrival (or birth) and departure (or death) rates.

Thus, a birth and death process is a continuous-time Markov chain with states $\{0, 1, \ldots\}$ For which transitions from state n may go only to either state $n-1$ or state $n+1$. The relationships between the birth and death rates and the state transition rates and probabilities are

$$\begin{aligned}
v_0 &= \lambda_0, \\
v_i &= \lambda_i + \mu_i, \qquad\qquad i > 0 \\
P_{01} &= 1, \\
P_{i,i+1} &= \frac{\lambda_i}{\lambda_i + \mu_i}, \qquad\qquad i > 0 \\
P_{i,i-1} &= \frac{\mu_i}{\lambda_i + \mu_i}, \qquad\qquad i > 0
\end{aligned}$$

Ex. (The Poisson Process): Consider a birth and death process for which

$$\begin{aligned}
\mu_n &= 0, \qquad \text{for all} \quad n \geq 0 \\
\lambda_n &= \lambda, \qquad \text{for all} \quad n \geq 0
\end{aligned}$$

This is a process in which departures never occur, and the time between successive arrivals is exponential with mean $1/\lambda$. Hence, this is just the Poisson process.

Def: A birth and death process for which $\mu_n = 0$ for all n is called a pure birth process.

Ex. (A Birth Process with Linear Birth Rate): Consider a population whose members can give birth to new members but cannot die. If each member acts independently of the others and takes an exponentially distributed amount of time, with mean $1/\lambda$, to give birth, then if $X(t)$ is the population size at time t, then $X(t), t \geq 0$ is a pure birth process with $\lambda_n = n\lambda, n \geq 0$. This follows since if the population consists of n persons and each gives birth at an exponential rate λ, then the total rate at which births occur is $n\lambda$. This pure birth process is known as a Yule process after G. Yule who used it in his mathematical theory of evolution.

Ex. (A Linear Growth Model with Immigration): A model which

$$\begin{aligned} \mu_n &= n\mu, & n &\geq 1 \\ \lambda_n &= n\lambda + \theta, & n &\geq 0 \end{aligned}$$

is called a linear growth process with immigration. Such processes occur naturally in the study of biological reproduction and population growth. Each individual in the population is assumed to give birth at an exponential rate λ; in addition, there is an exponential rate of increase θ of the population due to an external source such as immigration. Hence, the total birth rate where there are n persons in the system is $n\lambda + \theta$. Deaths are assumed to occur at an exponential rate μ for each member of the population, and hence $\mu_n = n\mu$.

Ex. (The Queueing System $M/M/1$): Suppose that customers arrive at a single-server service station in accordance with a Poisson process having rate λ. That is, the times between successive arrivals are independent exponential random variables having mean $1/\lambda$.

Upon arrival, each customer goes directly into service if the server is free; if not, then the customer joins the queue (that is, he waits in line). When the server finishes serving a customer, the customer leaves the system and the next customer in line, if there are any waiting, enters the service. The successive service times are assumed to be independent exponential random variables having mean $1/\mu$.

The preceding is known as the $M/M//1$ queueing system. The first M refers to the fact that the interarrival process is Markovian (since it is a Poisson process) and the second to the fact that the service distribution is exponential (and, hence, Markovian). The 1 refers to the fact that there is a single server.

If we let $X(t)$ denote the number in the system at time t then $X(t), t \geq 0$ is a birth and death process with

$$\mu_n = \mu, \quad n \geq 1$$
$$\lambda_n = \lambda, \quad n \geq 0.$$

Ex. (A Multiserver Exponential Queueing System): Consider an exponential queueing system in which there are s servers available. An entering customer first waits in line and then goes to the first free server. This is a birth and death process with parameters

$$\mu_n = \begin{cases} n\mu, & 1 \leq n \leq s \\ s\mu, & n > s \end{cases}$$
$$\lambda_n = \lambda, \qquad n \geq 0$$

To see why this is true, reason as follows: If there are n customers in the system, where $n \leq s$, then n servers will be busy. Since each of these servers works at a rate μ, the total departure rate will be $n\mu$. On the other hand, if there are n customers in the system, where $n > S$, then all s of the servers will be busy, and thus the total departure rate will be $s\mu$. This is known as an $M/M/s$ queueing model (why?).

Basic Properties for $P_{ij}(t)$

Let $q_{ij} = \nu_i P_{ij}$ be the *instantaneous transition rates*.

- It holds that

$$P_{ii}(h) = 1 - \nu_i h + o(h), \quad P_{ij}(h) = q_{ij} h + o(h), \quad i \neq j.$$

- Chapman-Kolmogorov equations: For all $s \geq 0$, $t \geq 0$,

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

- The Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j}^{\infty} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t), \quad i, j \geq 0.$$

- The Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i}^{\infty} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t), \quad i, j \geq 0.$$

In particular, for pure birth process,

$$P'_{i,j}(t) = \lambda_i P_{i+1,j}(t) - \lambda_i P_{ij}(t), \quad i, j \geq 0.$$

Limiting Probabilities for Continuous-time MC

For an irreducible, positive recurrent continuous-time Markov Chain $X(t), t \geq 0$ with parameters ν_j, P_{ij} and $q_{ij} = \nu_i P_{ij}$,

$$\lim_{t\to\infty} P_{ij}(t) = \lim_{t\to\infty} \mathbb{P}(X(t+s) = j \,|\, X(s) = i)$$

exists and is independent of i. Furthermore, letting

$$P_j = \lim_{t\to\infty} P_{ij}(t)$$

then P_j is the unique non-negative solution of

$$\nu_j P_j = \sum_{k\neq j}^{\infty} q_{kj} P_k, \quad j \geq 0$$

with

$$\sum_{j=0}^{\infty} P_j = 1.$$

- The equation follows from the set of forward equations

$$P'_{ij}(t) = \sum_{k\neq j}^{\infty} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t) \quad j \geq 0$$

by taking $t \to \infty$.

- P_j is the probability that the process is in state j after a long time.

- P_j is the long run proportion of time that the process is in state j (or visit j).

- P_j is the stationary distribution (probabilities).

- ν_j is the rate of leaving j given the process is in j.

- $\nu_j P_j$ is the rate at which the process leaves j.

- $q_{ij} = \nu_i P_{ij}$ is the *instantaneous transition rates* and determines the MC since

$$\nu_i = \sum_j q_{ij}, \qquad P_{ij} = q_{ij} \Big/ \sum_j q_{ij}.$$

- It holds that

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = \nu_i, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}, \quad i \neq j.$$

Embedded Chain and Time Reversibility

Consider the an ergodic continuous-time MC $\{X(t)\}$ with parameters ν_i and P_{ij}, and its embedded M.C. Y_n with P_{ij}. Let π_i be the limiting probabilities for Y_n, i.e.

$$\pi_i = \sum_j \pi_j P_{ji}, \quad \text{for all } i, \sum_i \pi_i = 1.$$

Then P_i should be in proportion to π_i/ν_i. Thus

$$P_i = \frac{\pi_i/\nu_i}{\sum_j \pi_j/\nu_j}.$$

- $X(t)$ is time reversible if and only if Y_n is time reversible.
- $X(t)$ is time reversible if and only if $P_i q_{ij} = P_j q_{ji}$ for all i, j.
- An ergodic birth and death process is time reversible.
- If there exist $x_i \geq 0$, $\sum x_i = 1$, such that

$$x_i q_{ij} = x_j q_{ji}, \quad \forall i \neq j,$$

then the continuous-time Markov chain is time reversible and $x_i = P_i$. This provides a way to find P_i.

HW 13. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean $\frac{1}{4}$ hour. What is
(a) the average number of customers in the shop?
(b) the proportion of potential customers that enter the shop?
(c) If the barber could work twice as fast, how many more business would he do?
Sol: We use three states 0, 1, and 2, representing number of customers in the shop. Then

$$\lambda_0 = 3, \quad \lambda_1 = 3, \quad \lambda_2 = 0$$
$$\mu_0 = 0, \quad \mu_1 = 4, \quad \mu_2 = 4$$

and hence $v_0 = 3$, $v_1 = 7$, $v_2 = 4$,

$$P_{01} = 1, \quad P_{12} = 3/7, \quad P_{10} = 4/7, \quad P_{21} = 1.$$

We can find $P_0 = \frac{16}{37}$, $P_1 = \frac{12}{37}$, $P_2 = \frac{9}{37}$.
(a) the average number is $0 \cdot P_0 + 1 \cdot P_2 + 2 \cdot P_2 = \frac{30}{37}$.
(b). $P_0 + P_1 = \frac{28}{37} \approx 0.7568$.
(c) We have $\mu_1 = \mu_2 = 8$ and $\bar{P}_0 + \bar{P}_1 = \frac{88}{97} \approx 0.9072$, about an additional 15%. Note that if we hire one more person, then $\mu_1 = 4$, $\mu_2 = 8$.

HW 14. Potential customers arrive at a full-service, one-pump gas station at a Poisson rate of 20 cars per hour. However, customers will only enter the station for gas if there are no more than two cars (including the one currently being attended to) at the pump. Suppose the amount of time required to service a car is exponentially distributed with a mean of five minutes.

(a) What fraction of the attendant's time will be spent servicing cars?

(b) What fraction of potential customers are lost?

Sol: We use four states 0, 1, 2, and 3, representing number of cars in the station. Then

$$\lambda_0 = 20, \quad \lambda_1 = 20, \quad \lambda_2 = 20, \quad \lambda_3 = 0$$
$$\mu_0 = 0, \quad \mu_1 = 12, \quad \mu_2 = 12, \quad \mu_3 = 12.$$

We can find $P_0 = \frac{27}{272}$, $P_1 = \frac{45}{272}$, $P_2 = \frac{75}{272}$, $P_3 = \frac{125}{272}$.

(a). $P_1 + P_2 + p_3 = 1 - P_0 = \frac{245}{272} \approx 0.9007.$

(b). $P_3 = \frac{125}{272} \approx 0.4596.$

•For a time-homogeneous Markov process $X(t), t \geq 0$, the subsequence $X(0), X(2\varepsilon), X(3\varepsilon), \cdots$ forms a Markov chain for any $\varepsilon > 0$ fixed. What is its transition matrix?

•Q-matrix:

$$\lim_{t \to 0} \frac{P_t - P_0}{t} = Q$$

•$\nu_i = -q_{i,i}$

•$P_t'|_{t=0} = Q$ and $P_t' = P_t Q$ and $P_t' = Q P_t$. The equations are called the (Kolmogorov) forward and backward equations (respectively).

•Just like $y' = cy$ has a solution $y = y_0 e^{ct}$, the Kolmogorov equation has the solution $P_t = e^{Qt}$ where the exponential of a matrix is defined to be

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

•P_t is a semigroup and Q is the generator.

Semigroup and Infinitesimal Generator

A simple example: Define for $t \geq 0$,

$$T_t(x) = \begin{cases} x - t & \text{if } x > t \\ x + t & \text{if } x < -t \\ 0 & \text{if } |x| \leq t \end{cases}$$

Then T_t is a semigroup on $\mathbb{R}$ with generator

$$A(x) = -\text{sgn}(x) = -\frac{x}{|x|}.$$

Pf: Write $T_t(x) = \text{sgn}(x)(|x| - t)^+$, $T_0(x) = x$.

•In physics, one-parameter groups describe dynamical systems, see Zeidler, E. Applied Functional Analysis: Main Principles and Their Applications. Springer-Verlag, 1995.

•Semigroup methods in partial differential equations: Semigroup theory can be used to study some problems in the study of partial differential equations. Roughly speaking, the semigroup approach is to regard a time-dependent partial differential equation as an ordinary differential equation on a function space.

Uniformization Method

Let $N(t)$ be the number of state transitions by time t. Suppose that $\nu_i = \nu$ for all states i. Then $\{N(t), t \geq 0\}$ is a Poisson process with rate ν, and

$$\begin{aligned} P_{ij}(t) &= \mathbb{P}(X(t) = j \mid X(0) = i) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(X(t) = j \mid X(0) = i, N(t) = n) \\ &\qquad \cdot \mathbb{P}(N(t) = n \mid X(0) = i) \\ &= \sum_{n=0}^{\infty} P_{ij}^{(n)} \cdot e^{-\nu t} \frac{(\nu t)^n}{n!} \end{aligned}$$

where

$$P_{ij}^{(n)} = \mathbb{P}(Y_n = j \mid Y_0 = i)$$

for the embedded Chain (Y_n, P_{ij}) with $P_{ii} = 0$.

- The above formula is useful from computation point of view via approximation by partial sum.

- What about $\nu_i \not\equiv \nu$ for all i but $\nu_i \leq \nu$ for all i? (Special case to general case).

We can think the chain with $\nu_i \le \nu$ for all i spends an exponential amount of time with rate ν in state i and then makes a transition to j with probability P^*_{ij}, where

$$P^*_{ij} = \begin{cases} 1 - \frac{\nu_i}{\nu} & \text{if } j = i \\ \\ \frac{\nu_i}{\nu} P_{ij} & \text{if } j \neq i. \end{cases}$$

Hence we obtain

$$P_{ij}(t) = \sum_{n=0}^{\infty} P^{*(n)}_{ij} \cdot e^{-\nu t} \frac{(\nu t)^n}{n!}$$

where $P^{*(n)}_{ij} = \mathbb{P}(U_n = j \,|\, U_0 = i)$ for the uniformized Chain (U_n, P^*_{ij}).

- We introduced transitions from a state to itself.
- We need the following fact: If e_i are i.i.d expo(ν) and

$$T^*_i = \begin{cases} e_1 & \text{w.p. } \frac{\nu_i}{\nu} \\ e_1 + e_2 & \text{w.p. } (1 - \frac{\nu_i}{\nu}) \cdot \frac{\nu_i}{\nu} \\ \cdots & \\ e_1 + e_2 + \cdots + e_n & \text{w.p. } (1 - \frac{\nu_i}{\nu})^{n-1} \cdot \frac{\nu_i}{\nu} \\ \cdots & \end{cases}$$

then $T^*_i \sim$ expo(ν_i).

Pf: Note that given i.i.d $e_j \sim$ expo(ν), $j \geq 1$,

$$T_i^* = \sum_{j=1}^{n} e_j \quad \text{w.p} \quad \left(1 - \frac{\nu_i}{\nu}\right)^{n-1} \cdot \frac{\nu_i}{\nu}.$$

Then for any $\lambda > 0$, the Laplace transform

$$\begin{aligned}
\mathbb{E}\, e^{-\lambda T_i^*} &= \mathbb{E} \sum_{n=1}^{\infty} e^{-\lambda \sum_{j=1}^{n} e_j} \cdot \left(1 - \frac{\nu_i}{\nu}\right)^{n-1} \cdot \frac{\nu_i}{\nu} \\
&= \frac{\nu_i}{\nu} \sum_{n=1}^{\infty} \left(1 - \frac{\nu_i}{\nu}\right)^{n-1} \cdot \mathbb{E}\, e^{-\lambda \sum_{j=1}^{n} e_j} \\
&= \frac{\nu_i}{\nu} \sum_{n=1}^{\infty} \left(1 - \frac{\nu_i}{\nu}\right)^{n-1} \cdot \prod_{j=1}^{n} \mathbb{E}\, e^{-\lambda e_j} \\
&= \frac{\nu_i}{\nu} \sum_{n=1}^{\infty} \left(1 - \frac{\nu_i}{\nu}\right)^{n-1} \cdot \left(\frac{\nu}{\lambda + \nu}\right)^{n} \\
&= \frac{\nu_i}{\lambda + \nu} \sum_{n=1}^{\infty} \left(\frac{\nu - \nu_i}{\lambda + \nu}\right)^{n-1} \\
&= \frac{\nu_i}{\lambda + \nu_i}
\end{aligned}$$

where

$$\mathbb{E}\, e^{-\lambda e_j} = \int_0^{\infty} e^{-\lambda x} \cdot \nu e^{-\nu x} dx = \frac{\nu}{\lambda + \nu}.$$