Solutions of EXAM I, Spring 2008

1. (12 pts). A coin, having probability p of landing heads and probability $q = 1 - p$ of landing tails, is continually flipped until at least two head and one tail have been flipped. Find the expected number of flips that lands on heads.

Sol 1: Let N be the number of flips that lands on heads when stopped. Then $\mathbb{E}\,N = \mathbb{E}\,(N|H) \cdot p + \mathbb{E}\,(N|T) \cdot q$ and $\mathbb{E}\,(N|H) = \mathbb{E}\,(N|HH) \cdot p + \mathbb{E}\,(N|HT) \cdot q$. Using geometric r.v. or additional conditioning, we obtain $\mathbb{E}\,(N|HH) = 2 + (\frac{1}{q} - 1) = 1 + \frac{1}{q}$. It is self-evident that $\mathbb{E}\,(N|HT) = 2$ and $\mathbb{E}\,(N|T) = 2$. Hence $\mathbb{E}\,(N|H) = q + \frac{1}{q}$ and

$$\mathbb{E}\,N = 2 + \frac{p^3}{q} = \frac{1}{q} + q - p^2 = \frac{p}{q} + pq + 2q = 1 + q + pq + \frac{p^2}{q} = ...$$

Sol 2: Conditioning on outcomes of the first two flips, $X = HH$ or $X^c = (HH)^c$. Then $\mathbb{E}\,N = \mathbb{E}\,(N|X) \cdot p^2 + \mathbb{E}\,(N|X^c) \cdot (1 - p^2)$, $\mathbb{E}\,(N|X) = 1 + 1/q$ and $\mathbb{E}\,(N|X^c) = 2$.

2. (18 pts). A Markov chain $X_0, X_1, X_2, \cdots$ has the transition probability matrix: $\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ and the initial distribution $\mathbb{P}(X_0 = 0) = 0.4, \mathbb{P}(X_0 = 1) = 0.2, \mathbb{P}(X_0 = 2) = 0.4$. Determine the following:

(a). Is the Markov chain time reversible? Why?

Ans: Yes since $\pi_i P_{ij} = \pi_j P_{ji}$, $\pi_0 = 1/4, \pi_1 = 1/2, \pi_2 = 1/4$.

(b). $\lim_{n\to\infty} \mathbb{P}(X_{n+4} = 2, X_{n+3} = 1, X_{n+2} = 0 \mid X_{n+1} = 0, X_n = 1)$. **Ans:** 0.

(c). $\lim_{n\to\infty} \mathbb{P}(X_{n+2} = 0, X_{n+1} = 1, X_n = 2 \mid X_2 = 0)$.

Ans: $P_{10} \cdot P_{21} \cdot \pi_2 = 1/32$.

(d). $\mathbb{P}(X_2 = 1 \mid X_0 = 2)$. **Ans:** $P_{21}^{(2)} = P_{20}P_{01} + P_{21}P_{11} + P_{22}P_{21} = 1/2$.

(e). $\lim_{n\to\infty} \mathbb{P}(X_n = 2, X_{n+1} = 0, X_{n+2} = 1)$.

Ans: $\pi_2 P_{20} P_{01} = 1/16$.

(f). $\lim_{n\to\infty} \mathbb{P}(X_n = 0 \mid X_{2n} = 1)$ **Ans:** $\pi_0 = 1/4$.

(g). The mean return time to state 2. **Ans:** $m_2 = 1/\pi_2 = 4$.

(h). The long run fraction of time the process spends in state 1.

Ans: $\pi_1 = 1/2$.

(i). Starting from state 1, what is the expected number of visit to state 2 before returning to state 1? **Ans:** $\pi_2 m_1 = \pi_2/\pi_1 = 1/2$.

3. (6 pts). Trials are performed in a sequence. If the last two trials were successes, then the next trial is a success with probability .8; otherwise the next trial is a success with probability .5. Define a Markov chain which will help you to determine the long run proportion of trials are successes. Give the one-step transition matrix but do NOT find the stationary distribution in order to save time.

Ans: Let X_i be the result of trial i (so, say $X_i = 0$ if the i^{th} trial is a success, and $X_i = 1$ otherwise). Because the distribution of X_n depends on both X_{n-1} and X_{n-2}, (note that X is not a Markov chain), one can study the "enlarged" 2-dimensional chain $Y = Y_1, Y_2, \ldots, Y_n, \ldots$ so that $Y_n = 0$ if $(X_{n-1}, X_n) = (0,0)$, $Y_n = 1$ if $(X_{n-1}, X_n) = (0,1)$, $Y_n = 2$ if $(X_{n-1}, X_n) = (1,0)$, and $Y_n = 3$ if $(X_{n-1}, X_n) = (1,1)$. Then Y is a Markov chain with transition matrix $\mathbf{P} = \begin{pmatrix} .8 & .2 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ .5 & .5 & 0 & 0 \\ 0 & 0 & .5 & .5 \end{pmatrix}$.

•In the long run, what proportion of trials are successes?

It is easy to check that Y is irreducible and aperiodic, and has stationary distribution given by $\pi_0 = 5/11$, $\pi_1 = 2/11$, $\pi_2 = 2/11$ and $\pi_3 = 2/11$. Therefore, the fraction of the trials that are successes is $\pi_0 + \pi_1 = 7/11$. This can also be calculated by $\pi_0 + \pi_2$ or $\pi_0 + \pi_1/2 + \pi_2/2$.

Sol 2: One can also set up a three state Markov Chain. The states are: 0=SS, 1=FS and 2=(SF or FF). The transition matrix is

$$\mathbf{P} = \begin{pmatrix} .8 & 0 & .2 \\ .5 & 0 & .5 \\ 0 & .5 & .5 \end{pmatrix}.$$

The proportion of successful trials is $\pi_0 + \pi_1$.

4. (14 pts). Next semester your professor will teach Math630 and Math850 and possess three textbooks for Math 630 and two textbooks for Math850. He will always keep two at home and three at the office. At the beginning of a workday, he will randomly take one of his two textbooks at home to the office. And at the end of the day, he will randomly take one of his four textbooks in the office to home.

(1). Define a Markov chain with 3 states which will help you to determine the proportion of days that your professor can't use his textbook for Math850 at home.

Ans: Define states as number of math850 textbooks at home at the end of the day. Then the transition matrix is $\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 5/8 & 1/8 \\ 0 & 3/4 & 1/4 \end{pmatrix}$.

and we can find by using $\pi_i P_{ij} = \pi_j P_{ji}$,

$$\pi_0 = 3/10, \quad \pi_1 = 3/5, \quad \pi_2 = 1/10.$$

(2). Is your Markov chain time reversible? Why?

Ans: Yes. There is a solution for $\pi_i P_{ij} = \pi_j P_{ji}$.

(3). What fraction of workdays your professor can't use his textbook for Math850 at home? **Ans:** $\pi_0 = 3/10$.