S8.1-2. Sol: (a). $\sum_{x=1}^{3}\sum_{y=1}^{2} c(x+y) = 1$ implies $c = 1/21$.

(b). $p_X(x) = \sum_{y=1}^{2}(1/21)(x+y) = (2x+3)/21$ for $x = 1,2,3$.
$p_Y(y) = \sum_{x=1}^{3}(1/21)(x+y) = (6+3y)/21$ for $y = 1,2$.

(c). $\mathbb{P}(X \geq 2|Y = 1) = \frac{\mathbb{P}(X\geq 2, Y=1)}{\mathbb{P}(Y=1)} = \frac{p(2,1)+p(3,1)}{p_Y(1)} = \frac{7/21}{9/21} = 7/9$

(d). $\mathbb{E}\,X = \sum_{x=1}^{3}\sum_{y=1}^{2} x\cdot(1/21)(x+y) = 46/21$ or $\mathbb{E}\,X = \sum_{x=1}^{3} xp_X(x) = \sum_{x=1}^{3} x(2x+2)/21 = 46/21$. Similarly, we have $\mathbb{E}\,Y = 11/7$.

S8.1-10. Sol: (a). The set of possible values of X is $[0,1]$. For $0 \leq x \leq 1$, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x 8xydy = 4x^3$ and thus

$$f_X(x) = \begin{cases} 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

Similarly, $f_Y(y) = \int_y^1 8xydx = 4y(1-y^2)$ for $0 \leq y \leq 1$.
(b). $\mathbb{E}\,(X) = \int_0^1 xf_X(x)dx = \int_0^1 x\cdot 4x^3dx = 4/5$;
$\int_0^1 yf_Y(y)dy = \int_0^1 y\cdot 4y(1-y^2)dy = 8/15$.

S8.2-12. Sol: Clearly,

$$\begin{aligned}\mathbb{E}(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dydx &= \int_0^1\int_x^1 (xy)(8xy)dydx \\ &= \int_0^1 \left(x^2\int_x^1 8y^2dy\right)dx=\frac{4}{9}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dydx=\int_0^1\int_x^1 x(8xy)dydx=\frac{8}{15}\\ \mathbb{E}(Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dydx=\int_0^1\int_x^1 y(8xy)dydx=\frac{4}{5}.\end{aligned}$$

So $\mathbb{E}(XY)\neq\mathbb{E}(X)\mathbb{E}(Y)$.

S8.2-20. Sol: Let X be the lifetime of the muffler from company A and Y be the lifetime of the muffler from company B. The joint probability density function of X and Y is $h(x,y)=f(x)g(y)$, for $x>0, y>0$ and 0 otherwise. So the desired probability is

$$\mathbb{P}(Y>X)=\int_0^{\infty}\left(\frac{1}{6}e^{-x/6}\int_x^{\infty}\frac{2}{11}e^{-2y/11}dy\right)dx=\frac{11}{23}.$$

S8.3-2: Sol: Since

$$f_Y(y) = \int_0^y 2dx = 2y, \quad 0 < y < 1,$$

we have that

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{2y} = \frac{1}{y}, \; 0 < x < y, 0 < y < 1.$$

S8.3-8: Sol: See Ex 8.22 in the textbook for a similar problem. Let $f(x,y)$ be the joint probability density function of X and Y. Clearly, $f(x,y) = f_{X|Y}(x|y) \cdot f_Y(y)$. Now $f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$ and

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{1-y} & y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus for $0 < x < 1$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X|Y}(x|y) \cdot f_Y(y) dy \\ &= \int_0^x (1-y)^{-1} dy = -\ln(1-x), \end{aligned}$$

and hence $f_X(x) = \begin{cases} -\ln(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

R8-2: Sol: (a) and (b). $p(x, y)$, the joint probability mass function of X and Y, and $p_X(x)$ and $p_Y(y)$, the marginal probability mass functions of X and Y can be found in a table form.

(c). $\mathbb{E}(X) = \sum_{x=2}^{15} x p_X(x) = 7$; $\mathbb{E}(Y) = \sum_{y=1}^{6} y p_Y(y) = 161/36 \approx 4.47$.

R8-6: Sol: (a).$\int_0^2 \left(\int_0^x \frac{c}{x} dy \right) dx = 1$ implies $c = 1/2$.

(b). $f_X(x) = \int_0^x \frac{1}{2x} dy = 1/2$, $0 < x, 1/2$. $f_Y(y) = \int_y^2 \frac{1}{2x} dx = \frac{1}{2} \ln(2/y)$, $0 < y < 2$.

S8.4–4: Sol: By the convolution theorem (Theorem 8.9), g, the probability density function of the sum of X and Y , the two random points selected from $(0, 1)$ is given by

$$g(t) = \int_{-\infty}^{\infty} f_1(x) f_2(t - x) dx$$

where f_1 and f_2 are, respectively, the probability density functions of X and Y. Since $f_1(x) = f_2(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$ the integrand $f_1(x)f_2(t - x)$ is nonzero if $0 < x < 1$ and $0 < t - x < 1$, that is $0 < x < 1$ and $t - 1 < x < t$. This shows that for $t < 0$ and $t > 2$, $g(t) = 0$. For $0 < t < 1$, $t - 1 < 0$; thus

$$g(t) = \int_0^t 1 dx = t.$$

For $1 < t < 2$, $0 < t - 1 < 1$; therefore,

$$g(t) = \int_{t-1}^{1} 1 dx = 1 - (t - 1) = 2 - t.$$

So $g(t) = \begin{cases} t & 0 < t < 1 \\ 2 - t & 1 < t < 2 \\ 0 & \text{otherwise,} \end{cases}$