S3.1-8: $\mathbb{P}(5|D) = \frac{\mathbb{P}(5 \text{ and } D)}{\mathbb{P}(D)} = \frac{4/36}{30/36} = 4/30$. The final answer can also be obtained by looking at the reduced sample space.

S3.1-18. A number is selected at random from the set $\{1, 2, \cdots, 10,000\}$ and is observed to be odd. What is the probability that it is (a) divisible by 3; (b) divisible by neither 3 nor 5.

Sol: (a). Let $T = \{\text{divisible by 3}\}$ and $O = \{\text{odd}\}$. Then

$$\mathbb{P}(T|O) = \frac{\mathbb{P}(TO)}{\mathbb{P}(O)} = \frac{1667}{5000} \neq \frac{3333}{10000}$$

since $\mathbb{P}(O) = 5000/10000$ and $\mathbb{P}(TO) = 1667/10000$. We can also find the answer by considering the reduced sample space $\{1, 3, \cdots, 9999\}$.

(b). Let $F = \{\text{divisible by 5}\}$. Then by using $\mathbb{P}(\bullet|O)$ is a probability measure, we have

$$\begin{aligned}
\mathbb{P}(T^cF^c|O) &= 1 - \mathbb{P}((T \cup F)|O) \\
&= 1 - (\mathbb{P}(T|O) + \mathbb{P}(F|O) - \mathbb{P}(TF|O)) \\
&= 1 - \frac{1667}{5000} - \frac{1000}{5000} + \frac{333}{5000}
\end{aligned}$$

Each term can be found as in (a) by using

$$\mathbb{P}(FO) = \frac{1000}{10000}, \quad \mathbb{P}(TFO) = \frac{333}{10000}.$$

S3.2-6: $\frac{3}{8} \times \frac{5}{10} \times \frac{5}{13} \times \frac{8}{15} + \frac{5}{8} \times \frac{3}{11} \times \frac{8}{13} \times \frac{5}{16}$.

S3.2-10. From an ordinary deck of 52 cards, cards are drawn one by one, at random and without replacement. What is the probability that the fourth heart is drawn on the tenth draw?

Hint: Let F denote the event that in the first nine draws there are exactly three hearts and E be the event that the tenth draw is a heart. Use $\mathbb{P}(FE) = \mathbb{P}(E|F)\mathbb{P}(F)$.

Sol: We have

$$\mathbb{P}(F) = \frac{\binom{13}{3}\binom{39}{6}}{\binom{52}{9}}$$

and

$$\mathbb{P}(E|F) = \frac{\binom{10}{1}\binom{33}{0}}{\binom{43}{1}} = \frac{10}{43}.$$

Key: Conditioning on what happens before the real thing!!

S3.3-10. **Sol 1:** Let B be the event that a randomly selected child from the countryside is a boy. Let I denote the event that the selected child is the first child and II denote the event that the selected child is the second child. Then by the laws of total probability

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B|I)\cdot\mathbb{P}(I)+\mathbb{P}(B|II)\cdot\mathbb{P}(II)\\ &= \frac{1}{2}\cdot\mathbb{P}(I)+\frac{1}{2}\cdot\mathbb{P}(II)=\frac{1}{2}.\end{aligned}$$

Note that we do not need to know $\mathbb{P}(I)$ but what is $\mathbb{P}(I)$? **Ans:** $\mathbb{P}(I) = 2/3$. Note also that $\mathbb{P}(B)$ different if we randomly pick a family first and then pick one of their child(ren).

What happens if they allow the couple to stop until they have a boy? **Ans:** $\mathbb{P}(B) = 1/2$.

Sol 2: Here is Marilyn Vos Savant's intuitive solution to this problem: *The distribution of sexes will remain roughly equal. Thats becauseno matter how many or how few children are born anywhere, anytime, with or without restriction half will be boys and half will be girls: Only the act of conception (not the government!) determines their sex. One can demonstrate this mathematically. (In this example, well assume that women with firstborn girls will always have a second child.)*

Lets say 100 women give birth, half to boys and half to girls. The half with boys must end their families. There are now 50 boys and 50 girls. The half with girls (50) give birth again, half to boys and half to girls. This adds 25 boys and 25 girls, so there are now 75 boys and 75 girls. Now all must end their families. So the result of the policy is that there will be fewer children in number, but the boy/girl ratio will not be affected.

Sol 3: Let B_1 be the event that the first child in the selected child's family is a boy and G_1 be the event that the first child in the selected child's family is a girl. Then by the laws of total probability

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B|B_1)\cdot\mathbb{P}(B_1)+\mathbb{P}(B|G_1)\cdot\mathbb{P}(G_1)\\ &= 1\cdot ???+???\cdot ???\end{aligned}$$

Note that $\mathbb{P}(B|G_1) \neq \frac{1}{2}$, why?

What is $\mathbb{P}(B_1)$? **Ans:**

$$\begin{aligned}\mathbb{P}(B_1) &= \mathbb{P}(B_1|I)\cdot\mathbb{P}(I)+\mathbb{P}(B_1|II)\cdot\mathbb{P}(II)\\ &= \frac{1}{2}\cdot\frac{2}{3}+0\cdot\mathbb{P}(II)=\frac{1}{3}.\end{aligned}$$

What is $\mathbb{P}(B|G_1)$? **Ans:** 1/4

S3.3-12: A Probabilistic Sol: Let A be the event that a randomly selected adult is married. Let M be the event that the randomly selected adult is a man, and let W be the event that the randomly selected adult is a woman. By the law of total probability,

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A|M)\mathbb{P}(M) + \mathbb{P}(A|W)\mathbb{P}(W) \\ &= \frac{7}{9} \cdot \mathbb{P}(M) + \frac{3}{5} \cdot \mathbb{P}(W)\end{aligned}$$

To find $\mathbb{P}(M)$, let n be the number of adults in the town and let x be the number of men in the town. Then $n - x$ is the number of women in the town. Since the number of married men and married women are equal, we have

$$x \cdot \frac{7}{9} = (n - x) \cdot \frac{3}{5}.$$

This relation implies that $x = (27/62)n$. Therefore, the probability that a randomly selected adult is male is $(27/62)n/n = 27/62$. The probability that a randomly selected adult is female is $1-(27/62) = 35/62$. Therefore, 21/31st of the adults are married.

An Arithmetical Sol: Following the second part of sol 1, the fraction of married adults are

$$\frac{x \cdot \frac{7}{9} + (n - x) \cdot \frac{3}{5}}{n} = \frac{21}{31}.$$

The Common Numerator Sol: The common numerator of the two fractions is 21. Hence 21/27th of the men and 21/35th of the women are married. We find the common numerator because the number of married men and the number of married women are equal. This shows that of every $27 + 35 = 62$ adults, $21 + 21 = 42$ are married. Hence 42/62th = 21/31st of the adults in the town are married.

S3.4-12. **Sol:** It is a Bayes problem!! Why?
Let T be the event that I start with a \$2 bill and W be the event that I start with a \$20 bill. Let II be the event that the removed bill is a \$2 bill. Then by Bayes' formula

$$\begin{aligned}
\mathbb{P}(T|II) &= \frac{\mathbb{P}(II|T)\cdot\mathbb{P}(T)}{\mathbb{P}(II|T)\cdot\mathbb{P}(T)+\mathbb{P}(II|W)\cdot\mathbb{P}(W)} \\
&= \frac{1\cdot\frac{1}{2}}{1\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}} = \frac{2}{3}.
\end{aligned}$$

S3.4-14: Let B be the event that the chip from the second urn is blue and B_i, $i = 0, 1, 2, 3$, be the events that exact i blue chips, i.e. $4 - i$ red chips, is transferred. Then

$$\mathbb{P}(B_2|B) = \frac{\mathbb{P}(B_2 B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|B_2)\mathbb{P}(B_2)}{\sum_{i=0}^{4}\mathbb{P}(B|B_i)\mathbb{P}(B_i)}$$

with $\mathbb{P}(B_i) = \frac{\binom{5}{4-i}\binom{3}{i}}{\binom{8}{4}}$ and $\mathbb{P}(B|B_i) = \frac{i}{4}$ for $i = 0, 1, 2, 3$. **Ans:** 4/7. The answer can also be obtained by reducing the sample space to 5 red and 2 blue chips. The probability is $\frac{\binom{5}{2}\binom{2}{1}}{\binom{7}{3}} = \frac{4}{7}$.

S3.5-10: The events are not mutually exclusive. We have $\mathbb{P}(\text{at least one 6}) = 1-\mathbb{P}(\text{no 6}) = 1-(5/6)^4$ and $\mathbb{P}(\text{at least one double 6}$ $1 - \mathbb{P}(\text{no double 6}) = 1 - (35/36)^{24}$.

S3.5-22: $1 - (5/6)^6$.

R3-14: The probability that the first urn was selected in the first place is $\frac{\frac{20}{45}\cdot\frac{1}{2}}{\frac{20}{45}\cdot\frac{1}{2}+\frac{10}{25}\cdot\frac{1}{2}} = \frac{10}{19}$. The desired probability is $\frac{20}{45}\cdot\frac{10}{19} + \frac{10}{25}\cdot\frac{9}{19} \approx$ 0.42.

R3-18: The odds are the same. Let L be the event that the couple is lucky and L_F be the event that they are luck on Friday. Then by the laws of total probability,

$$\begin{aligned}
\mathbb{P}(L) &= \mathbb{P}(L|L_F)\cdot\mathbb{P}(L_F) + \mathbb{P}(L|L_F^c)\cdot\mathbb{P}(L_F^c) \\
&= 1\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{2}{3} = \frac{2}{3}.
\end{aligned}$$