S10.2-10:

Sol: (a). The result follows from

$$\begin{aligned} & \mathrm{Cov}(X-Y, X+Y) \\ = \ & \mathbb{E}\,((X-Y)(X+Y)) - (\mathbb{E}\,(X-Y)) \cdot (\mathbb{E}\,(X+Y)) \\ = \ & \mathbb{E}\,(X^2 - Y^2) - (\mathbb{E}\,X - \mathbb{E}\,Y) \cdot (\mathbb{E}\,X + \mathbb{E}\,Y) \\ = \ & \mathbb{E}\,(X^2) - \mathbb{E}\,(Y^2) - ((\mathbb{E}\,X)^2 - (\mathbb{E}\,Y)^2) \\ = \ & (\mathbb{E}\,(X^2) - (\mathbb{E}\,X)^2) - (\mathbb{E}\,(Y^2) - (\mathbb{E}\,Y)^2) \\ = \ & \mathrm{Var}(X) - \mathrm{Var}(Y). \end{aligned}$$

Note that independent assumption between X and Y is not used.

(b). We have

$$\begin{aligned} \mathrm{Cov}(X, XY) &= \mathbb{E}\,(X^2Y) - \mathbb{E}\,X \cdot \mathbb{E}\,(XY) \\ &= \mathbb{E}\,(X^2) \cdot \mathbb{E}\,Y - (\mathbb{E}\,X)^2 \cdot \mathbb{E}\,(Y) \\ &= \mathbb{E}\,Y \cdot (\mathbb{E}\,(X^2) - (\mathbb{E}\,X)^2) \\ &= \mathbb{E}\,(Y)\mathrm{Var}(X). \end{aligned}$$

S10.2-12: **Sol:** The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that for $-1 < x < 1$,

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi}$$

and for $-1 < y < 1$,

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2\sqrt{1-y^2}}{\pi}.$$

So X and Y are dependent because $f(x,y) \neq f_X(x) \cdot f_Y(y)$.

On the other hand, we have

$$\mathbb{E}(X) = \iint_{x^2+y^2\leq 1} x \cdot \frac{1}{\pi} dxdy = \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{x}{\pi} dx \right) dy = 0$$

and similarly

$$\begin{aligned} \mathbb{E}(Y) &= \iint_{x^2+y^2\leq 1} y \cdot \frac{1}{\pi} dxdy = 0 \\ \mathbb{E}(XY) &= \iint_{x^2+y^2\leq 1} xy \cdot \frac{1}{\pi} dxdy = 0. \end{aligned}$$

So $\text{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y) = 0$ and X and Y are uncorrelated.

S10.3-2: Sol: Note that X and Y are independent random variables. This can also be shown directly by verifying the relation $f(x,y) = f_X(x)f_Y(y)$. Hence Cov$(X,Y) = 0$, and therefore $\rho(X,Y) = 0$. You can also compute $\mathbb{E}\,X$, $\mathbb{E}\,Y$ and $\mathbb{E}\,(XY)$, and find they all equal to one. Hence Cov$(X,Y) = \mathbb{E}\,XY - \mathbb{E}\,X \cdot \mathbb{E}\,Y = 0$.

R10-12: Sol: (a). Clearly,

$$\begin{aligned} f_X(x) &= \int_0^x e^{-x} dy = xe^{-x}, \quad 0 < x < \infty, \\ f_Y(y) &= \int_y^\infty e^{-x} dx = e{-}y, \quad 0 < y < \infty. \end{aligned}$$

(b). We have that

$$\begin{aligned} \mathbb{E}\,(X) &= \int_0^\infty x^2 e^{-x} dx = 2, \quad \mathbb{E}\,(Y) = \int_0^\infty ye^{-y} dy = 1, \\ \mathbb{E}\,(X^2) &= \int_0^\infty x^3 e^{-x} dx = 6, \quad \mathbb{E}\,(Y^2) = \int_0^\infty y^2 e^{-y} dy = 2. \end{aligned}$$

Therefore, Var$(X) = 2$ and Var$(Y) = 1$. Also

$$\mathbb{E}\,(XY) = \int_0^\infty \left(\int_y^\infty xy \cdot e^{-x} dx \right) dy = 3.$$

Thus $\rho(X,Y) = \frac{\mathbb{E}\,(XY) - \mathbb{E}\,(X)\mathbb{E}\,(Y)}{\sigma_X \sigma_Y} = \frac{1}{\sqrt{2}}$.

Sol. of S11.5-2: For $1 \le i \le 35$, let X_i be the score of the i-th student selected at random. By the central limit theorem

$$
\begin{aligned}
& \mathbb{P}(460 < \bar{X} < 540) \\
= \; & \mathbb{P}(460 < \frac{X_1 + X_2 + \cdots + X_{35}}{35} < 540) \\
= \; & \mathbb{P}(16100 < X_1 + X_2 + \cdots + X_{35} < 18900) \\
= \; & \mathbb{P}(\frac{16100 - 35 \cdot 500}{100\sqrt{35}} < \frac{X_1 + X_2 + \cdots + X_{35}}{100\sqrt{35}} < \frac{18900 - 35 \cdot 500}{100\sqrt{35}}) \\
\approx \; & \mathbb{P}(-2.37 < Z < 2.37) = \Phi((2.37) - \Phi(-2.37) = 0.9822.
\end{aligned}
$$

Sol. of S11-R18: For $1 \le i \le 20$, let X_i denote the outcome of the i-th roll. We have

$$
\mathbb{E}\,(X_i) = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{7}{2}, \quad \mathbb{E}\,(X_i^2) = \sum_{i=1}^{6} i^2 \cdot \frac{1}{6} = \frac{91}{6}.
$$

Thus Var$(X_i) = (91/6) - (7/2)^2 = 35/12$, and hence

$$
\begin{aligned}
& \mathbb{P}(65 \le \sum_{i=1}^{20} X_i \le 75) \\
= \; & \mathbb{P}(\frac{65 - 70}{\sqrt{35/12} \cdot \sqrt{20}} \le \frac{\sum_{i=1}^{20} X_i - 70}{\sqrt{35/12} \cdot \sqrt{20}} \le \frac{75 - 70}{\sqrt{35/12} \cdot \sqrt{20}}) \\
\approx \; & \mathbb{P}(-0.65 \le Z \le 0.65) = \Phi(0.65) - \Phi(-0.65) = 0.4844.
\end{aligned}
$$