Moment Generating Function

Def: The Moment Generating Function (MGF) of a r.v. X is

$$M(t) = M_X(t) = \mathbb{E}\, e^{tX}, \quad t \in \mathbb{R}.$$

Basic properties of MGF:

• $\mathbb{E}\, X^n = M^{(n)}(0)$ and $\mathrm{Var}(X) = M''(0) - (M'(0))^2$.

• If X and Y are ind. with MGF $M_X(t)$ and $M_Y(t)$, respectively, then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

Ex 11.1: Determine $M_X(t)$ and $\mathbb{E}\,(X^n)$ for Bernoulli r.v with parameter p.

HW: S11.1: 1, **4**, 5; S11.5: **2**, 3, 7, R11: 1, 11, **18**.

Law of Large Numbers (LLN)

Thm. 11.10: (Weak Law of Large Numbers). Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed r.v's with $\mathbb{E}\, X_i = \mu$. Then for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right) = 0$$

Pf: Use Chebyshev's Inequality.

Thm. 11.11: (Strong Law of Large Numbers). Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed r.v's with $\mathbb{E}\, X_i = \mu$. Then

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu\right) = 1.$$

The Central Limit Theorem (CLT)

Thm. 3.1: (Central Limit Theorem). Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed r.v's with $\mathbb{E}\, X_i = \mu$ and $\sigma^2 = \text{Var}(X_i)$. Then, as $n \to \infty$,

$$\begin{aligned} \mathbb{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le x\right) &\to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \\ &= \Phi(x) = \mathbb{P}(Z \le x) \end{aligned}$$

where $Z \sim N(0, 1)$.

Idea of Pf: Show $M_{Z_n}(t) \to M_Z(t)$ for

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - \mathbb{E}\, S_n}{\sqrt{\text{Var}(S_n)}}.$$

where $S_n = X_1 + \cdots + X_n$.

• Let $\bar{X}_n = (X_1 + \cdots + X_n)/n$ denote the sample mean. Then $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$ is approximately standard normal, i.e

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \approx Z.$$

Ex 11.27: The lifetime of a TV tube (in years) is an exponential random variable with mean 10. What is the probability that the average lifetime of a random sample of 36 TV tubes is at least 10.5?

Ex 11.29: If 20 random numbers are selected independently from the interval $(0, 1)$, what is the approximate probability that the sum of these numbers is at least eight?

•Additional questions: At most 11? Between 8 and 13? More than 12 or less than 8?

Simulation

All of simulation starts with the question "how do I choose a random number uniformly between 0 and 1?" This is an intricate question, and most time we use pseudor-random numbers. These days, any self-respecting programming language or environment has a routine for this task (typically something like random[], rand, rnd, or some other variant therefrom). In this class, we will use such random number generators to generate a few other random variables of interest.

• Generating a Bernolli Random Variable X. If U is uniformly distributed on the interval $[0, 1]$, then it follows that $\mathbb{P}(U \leq p) = p$. So, if we defined

$$X := \begin{cases} 1 & \text{if} \quad U \leq p \\ 0 & \text{if} \quad U > p \end{cases}$$

then $\mathbb{P}(X = 1) = \mathbb{P}(U \leq p) = p$ and $\mathbb{P}(X = 0) = \mathbb{P}(U > p) = 1 - p$.

- Generating Discrete Random Variables. Assume $\mathbb{P}(X = x_i) = p_i$, $i = 0, 1, \cdots$, with $p_0 + p_1 + \cdots = 1$. Let U be uniform on $[0, 1]$. Define

$$X := \begin{cases} x_0 & \text{if} \quad U \le p_0 \\ x_1 & \text{if} \quad p_0 < U \le p_0 + p_1 \\ x_2 & \text{if} \quad p_0 + p_1 < U \le p_0 + p_1 + p_2 \\ \cdots & \cdots \end{cases}$$

- The Inverse Transform Method.

- Monte Carlo Method

- Rejection Method