Math 350: Probability Theory and Simulation (S09)

Professor: Wenbo Li (Office: 517b Ewing, Ph: 831-1865, email: wli@math.udel.edu).

Office Hours: 12:30-2:00 Tu, 2:00-3:30 Th, or by appointment.

Textbook: *Fundamentals of Probability, 3rd edition* by Saeed Ghahramani.

Prerequisites: PREREQ: MATH210 or MATH230; PREREQ or COREQ: MATH243

Course description: Elements of probability; Continuous, discrete, and joint distributions of random variables; Conditional probability; Functions of random variables; Expectation and conditional expectation; Law of large numbers and the central limit theorem; Sampling distributions; Simulation.

Homework: Homework will be assigned in class and some problems of even (**bold**) numbers will be discussed in class and used in quiz.

Quizzes: A total of at least 8 quizzes (10-15 minutes each) will be given in class on almost every Tuesday.

Grading: Grades will be based on 500 points (100 points for each in-class midterm, 160 points for the comprehensive final exam, 140 points for quizzes). One lowest quiz grade will be dropped. If you miss a midterm exam with legal reasons (need proofs), the final exam will count for that exam also. Tentative dates for midterm exam are March 17 and May 5.

Approximately course grade: A: 100-89%; B: 88-77%; C: 76-65%; D: 64-53%

Academic Honesty:

http://www.udel.edu/stuguide/07-08/code.html

Bertrand's paradox

Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of an equilateral triangle inscribed in a circle?
This problem was originally posed by Joseph Bertrand in his work, Calcul des probabilits (1888). Bertrand gave three arguments, all apparently valid, yet yielding inconsistent results.

1. The "random endpoints" method: Choose a point on the circumference and rotate the triangle so that the point is at one vertex. The probability is $1/3$.

2. The "random radius" method: Choose a radius of the circle and rotate the triangle so a side is perpendicular to the radius. The probability is $1/2$.

3. The "random midpoint" method: Choose a point anywhere within the circle and construct a chord with the chosen point as its midpoint. The probability is $1/4$.
•See http://en.wikipedia.org/wiki/Bertrand'sparadox(probability)
•This example shows that we must carefully define what we mean by the term "random".

Ch I: Axioms of Probability

The **sample space**, denoted by S, is the set of all possible outcomes in an "experiments". These outcomes are called **sample points** or simply **points**. An **event** is a subset of S.

Ex: Toss a coin twice.

Ex: The number of phone calls to math. dept. in a given day.

Ex: The length of time between successive earthquakes.

Ex: Arrival time and number of passengers for a bus.

- We can have finite, infinite or continuous sample space.

HW: S1.2: 1, 3, 5, 7, **10**, 11, 13; S1.4: 1, 3, **6**, 7, 9, 15, **18**, 19, 21, **22**, 25; S1.7: 1, 5, 7; R1: 3, **10**, 11, **14**, 15, 17.

HW: S2.2: 5, 7, 9, 11, 17, **18**, 19; S2.3: 1, 5, 11, **12**, 13; S2.4: 5, 13, 15, **18**, 19, 21, **30**, 33; R2: 3, 11, 15, 17, 21, **22**, **24**.

Basic Set Notions – Brief Review

- $\emptyset$: empty set.
- $\omega \in A$: ω is an element of the set A, or ω belongs to A.
- $\omega \notin A$: ω does not belong to A.
- $A \subset B$ or $B \supset A$: every element of A is an element of B; or A is a subset of B; or A is contained in B.
- $A = B$: $A \subset B$ and $B \subset A$.
- Note that $\subset$ or $\supset$ for sets and $\in$ or $\notin$ for elements.

Basic Set Operations – Brief Review

- Intersection: $AB = A \cap B = \{\omega : \omega \in A \,\text{and}\, \omega \in B\}$.
- Union: $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$.
- Difference: $A - B = \{\omega : \omega \in A \text{ and } \omega \notin B\}$.
- Complement: $A^c = S - A$ if S is the whole space.
- Cartesian product: $A \times B = \{(a, b) : a \in A,\, b \in B\}$.
- A and B are disjoint (exclusive) if: $A \cap B = \emptyset$.
- $(A^c)^c = A$.
- Partition of a set S: a collection of subsets of S which are pairwise disjoint (exclusive) and whose union is the entire set S.

Basic Facts of Set Operations

- Commutative laws: $E \cup F = F \cup E$ and $EF = FE$.

- Associative laws: $(E \cup F) \cup G = E \cup (F \cup G)$ and $(EF)G = E(FG)$.

- Distributive laws: $(E \cup F)G = (EG) \cup (FG)$ and $(EF) \cup G = (E \cup G)(F \cup G)$.

- DeMorgan's laws: For any index set I,

$$(\cup_{i \in I} E_i)^c = \cap_{i \in I} E_i^c, \quad (\cap_{i \in I} E_i)^c = \cup_{i \in I} E_i^c$$

- Basic idea of proofs: Elementwise method.

- A useful relation between two events E and F:

$$E = (EF) \cup (EF^c).$$

Axioms of Probability

A **probability** $\mathbb{P}$ or **probability measure** $\mathbb{P}$ on a sample space S is a function on events of S such that

(i). $\mathbb{P}(A) \geq 0$ for any event $A \subset S$;

(ii). $\mathbb{P}(S) = 1$;

(iii). For mutually exclusive events $A_1, A_2, \cdots$, (i.e $A_i \cap A_j = \emptyset$, $i \neq j$), $\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

Basic properties of $\mathbb{P}$:

- $\mathbb{P}(\emptyset) = 0$; $0 \leq \mathbb{P}(A) \leq 1$.
- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
- If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$.
- Proofs: Disjoint unions.

Inclusion-exclusion Formula

For any events E_i, $1 \le i \le n$,

$$\begin{aligned}\mathbb{P}(\cup_{i=1}^n E_i) &= \sum_{i=1}^n \mathbb{P}(E_i) - \sum_{i_1<i_2} \mathbb{P}(E_{i_1}E_{i_2}) + \cdots \\ &\quad +(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} \mathbb{P}(E_{i_1}E_{i_2}\cdots E_{i_r}) \\ &\quad + \cdots + (-1)^{n+1}\mathbb{P}(E_1E_2\cdots E_n)\end{aligned}$$

where the summation

$$\sum_{i_1<i_2<\cdots<i_r} \mathbb{P}(E_{i_1}E_{i_2}\cdots E_{i_r})$$

is taken over all of the $\binom{n}{r}$ possible subsets of size r of the set $\{1, 2, \cdots n\}$.

- Proof: Mathematical induction.

- For three events E_1, E_2 and E_3,

$$\begin{aligned}\mathbb{P}(E_1 \cup E_2 \cup E_3) &= \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3) \\ &\quad -\mathbb{P}(E_1E_2) - \mathbb{P}(E_1E_3) - \mathbb{P}(E_2E_3) + \mathbb{P}(E_1E_2E_3).\end{aligned}$$

Sample spaces having equally likely outcomes

Consider the model $S = \{1, 2, \cdots, N\}$ and define

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{N\}).$$

If $\mathbb{P}$ is a probability, then $\mathbb{P}(\{i\}) = 1/N$, for $i = 1, 2, \cdots, N$, and

$$\mathbb{P}(A) = \frac{\text{number of points in } A}{\text{number of points in } S} = \frac{|A|}{|S|} = \frac{|A|}{N}$$

Ex: If two dice are rolled, what is the probability that the sum of the up-turned faces will equal 7?

Ex. 1.6: A number is chosen at random from the set of integers $\{1, 2, \cdots, 1000\}$. What is the probability that it is divisible by 3 or 5?

Sol: Let A be the event that the outcome is divisible by 3 and B be the event that the outcome is divisible by 5. Then the desired probability is

$$\mathbb{P}(A \cup B) = ...$$

Ans: 0.467.

Ex: (Scientific American, 1978). Roll two fair and "standard" dice with values $\{1, 2, 3, 4, 5, 6\}$ and let X be the sum of the face-up values. Next roll two fair but "non-standard" dice with values $\{1, 2, 2, 3, 3, 4\}$ on one and $\{1, 3, 4, 5, 6, 8\}$ on the other, and let Y be the sum of the face-up values. Then $\mathbb{P}(X = k) = \mathbb{P}(Y = k)$ for all $k = 2, 3, \cdots, 12$.

Uniqueness: $(x+x^2+x^3+x^4+x^5+x^6)^2 = (x+2x^2+2x^3+x^4)(x+x^3+x^4+x^5+x^6+x^8)$.

Def: The **odds** of an event A is $\mathbb{P}(A) : \mathbb{P}(A^c)$ or $\mathbb{P}(A)$ to $\mathbb{P}(A^c)$. Thus we say the odds (in favor) of an event A are r to s if $\mathbb{P}(A) = r/(r+s)$. Similarly, the odds against an event A are r to s if $\mathbb{P}(A) = s/(r+s)$.

Ex: In drawing a card at random from an ordinary deck of 52 cards, the odds of an ace are 4 to 48 or, equivalently, 1 to 12. The probability of drawing an ace is $1/13$.

Ex: If two cards are dealt at random in a poker game, what are the odds of two aces?

Basic Counting – Brief Review

The multiplication principle: If one experiment has m outcomes and other has n outcomes, then there are $m \cdot n$ possible outcomes for the two experiments together.

Ex: Toss a coin twice.

The generalized multiplication principle: If the ith experiment, $1 \leq i \leq r$ has n_i outcomes, then there are $\prod_{i=1}^{r} n_i = n_1 \cdot n_2 \cdots n_r$ possible outcomes for the r experiments together.

Ex: Toss a coin ten times.

Ex: Number of ways from A to D.

Ex: How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Ex: How many without duplicated letters or numbers?

Permutations

An n-element permutation of a set with n objects is simply called a permutation,

• There are $n! = 1 \cdot 2 \cdots (n-1) \cdot n$ different permutations (ordered arrangement) of the n objects. Note that $0! = 1$ (why?). What is $(1/2)!$?

• The total number of different permutations of n objects, of which n_i, $1 \leq i \leq r$ are alike, is $\frac{n!}{n_1! \cdots n_r!}$.

Ex: How many different letter arrangements can be formed using the letters $PEPPER$?

Ex: How many different signals, each consisting of 9 flags, hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

Combinations

The total number of ways (ordered or unordered) to choose r objects (with replacement or without replacement) from n objects are:

- n^r if the selection is ordered and with replacement
- $n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$ if the selection is ordered and without replacement
- $\frac{n!}{(n-r)!r!} = \binom{n}{r}$ if the selection is unordered and without replacement
- $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$ if the selection is unordered and with replacement

Ex: From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? what if 2 of the men are feuding and refuse to serve on the committee together?

- Two ways to solve the problem. One is by direct counting. The other is by complement (a nice idea). Note also the difference between addition and multiplication.

Ex: Choose five students from 10 in a class. Then

(i). the number of ways to form a team is ??

(ii). the number of ways to form a line is ??

The binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Proof by induction
- Proof by counting

Ex: If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the drawn balls is white and the other two black?

Sol 1: (Based on ordered point of view). Consider ordered possible outcomes as our sample space.

Sol 2: (Based on unordered point of view). Consider unordered possible outcomes as our sample space.

- Which is better?

Ex: A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Sol: Unordered.

Ex 2.6: What is the probability that at least two students of a class of size n have the same birthday?

Sol: Use complement.

Ex: An urn contains n balls, of which one is special. If k of these balls are withdrawn one at a time, with each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

Sol 1: (Based on globe point of view). Let E be the event that the special ball is selected.

Sol 2: (Based on sequential point of view). Let E_i be the event that the special ball is the i-th ball to be chosen, $1 \leq i \leq k$.

- Which is better? What about m special balls with $m \leq k$?

Probabilities in Poker Hands

(see Exercises 20 in Section 2.4)

Hand	# of the hand	≈ Prob.
Str. Flush	$\binom{10}{1}\binom{4}{1} = 40$	.000015
4-of-a-Kind	$\binom{13}{1}\binom{4}{4}\binom{48}{1} = 624$	.00024
Full House	$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3,744$	.0014
Flush	$\binom{4}{1}\binom{13}{5} - 40 = 5,108$	.0020
Straight	$\binom{10}{1}\binom{4}{1}^5 - 40 = 10,200$	.0039
3-of-a-Kind	$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 54,912$	.0211
Two Pair	$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123,552$	.0475
One Pair	$\binom{13}{1}\binom{4}{2}\binom{12}{33}\binom{4}{1}^3 = 1,098,240$	.4226
Trash	$\binom{52}{5} - \text{sum of above} = 1,302,540$	.5012
Total	$\binom{52}{5} = 2,598,960$	

Lottery: PowerBall

The current rules for the interstate lottery, powerball, offered in Delaware are that a player must predict the outcome of 5 balls drawn from a set of 53 white balls and 1 ball drawn from a set of 42 red balls. The number of combinations possible are $\binom{53}{5}\binom{42}{1} = 120,526,770$.

The chances of predicting the winning combination are smaller than being dealt a royal flush from a fifty two card poker deck and then tossing a fair coin seven times and getting all heads, an event which has one out of $83,166,720$.

You are more likely to toss 26 fair coins in the air and have them all land heads (1 out of $67,108,864$) than you are to predict the winning combination.

If I divide a 3 square mile area into 1 foot square patches and hide 100 million dollars there, you are more likely guess where the money is (1 out of $83,635,200$) than to predict the winning lottery combination.

Conditional Probability

Definition: If $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$$

- Reduced sample space

- $\mathbb{P}(\bullet|B)$ is a probability measure

- The multiplication rule: $\mathbb{P}(AB) = \mathbb{P}(B) \cdot \mathbb{P}(A|B)$.

Ex: There are 10 white balls, 5 yellow balls and 10 black balls in an urn. One ball is selected at random and it is not black. What is the probability that it is yellow?
Sol 1: Via definition.
Sol 2: Via reduced sample space.

HW: S3.1: 3, 7, **8**, 9, **18**; S3.2: 3, **6**, 7, **10**; S3.3: 1, 5, 9, **10**, **12**; S3.4: 3, 5, 7, **12**, **14**; S3.5: 7, 9, **10**, 17, **22**; R: 3, 9, **14**, 15, **18**.

Ex: For a random selected family with two children, what is the probability that the other child of the family is a girl, given that at least one is a girl.

Ex: Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls are red?

Sol 1: Based on globe point of view.

Sol 2: Based on sequential point of view. Let R_i be the event that the i-th ball selected is red, $i = 1, 2$.

Laws of Multiplication

When a sequence of events $A_1, A_2, \cdots, A_n$ occur in time or some logical order, we can usually easily find find the probability

$$\mathbb{P}(A_1 A_2 \cdots A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1A_2)\cdots\mathbb{P}(A_n|A_1A_2\cdots A_{n-1}).$$

Ex: (See also **S3.2-10**). From an ordinary deck of 52 cards, cards are drawn one by one, at random and without replacement. What is the probability that the fourth heart is drawn on the tenth draw and the third space is drawn on the eleventh draw?

Sol: Let A_1 denote the event that in the first nine draws there are exactly three hearts and two spaces and A_2 (A_3) be the event that the tenth (eleventh) draw is a heart (space). Then

$$\begin{aligned}
\mathbb{P}(A_1A_2A_3) &= \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1A_2) \\
&= \frac{\binom{13}{3}\binom{13}{2}\binom{26}{4}}{\binom{52}{9}} \cdot \frac{\binom{10}{1}\binom{33}{0}}{\binom{43}{1}} \cdot \frac{\binom{11}{1}\binom{31}{0}}{\binom{42}{1}} \\
&= \frac{507}{5593} \cdot \frac{10}{43} \cdot \frac{11}{42} \approx 0.0055
\end{aligned}$$

Laws of Total Probability

Let B_1, $B_2, \cdots, B_n$ be a partition of the sample space Ω, i.e. $\cup_{i=1}^{n} B_i = \Omega$, $B_i B_j = \emptyset$ for $i \neq j$. Then

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i).$$

In particular,

$$\mathbb{P}(A) = \mathbb{P}(A|B) \cdot \mathbb{P}(B) + \mathbb{P}(A|B^c) \cdot \mathbb{P}(B^c).$$

Ex:(conti. from the last Ex.) What is the probability that the second ball selected is red?

Ex: There are 3 red balls and 2 white balls in urn A, and 2 red balls and 5 white ball in urn B. A fair coin is tossed. If it is head, a ball is draw from urn A, and if it is tail, a ball is drawn from run B.

(a). What is the the probability that a red ball is drawn?

(b). If a red ball is drawn, what is the probability that the coin is head up?

Ex 3.15 Suppose that 80% of the seniors, 70% of the juniors, 50% sophomores, and 30% of the freshmen of a college use the library of their campus frequently. If 30% of all students are freshmen, 25% are sophomores, 25% are juniors, and 20% are senors, what percentage of all students use the library frequently?

Sol: Let U be the event that a randomly selected student is using the library frequently. Let F, O, J, E be the events that he or she is a freshmen, sophomore, junior, or senor, respectively. Then

$$\mathbb{P}(U) = ...$$

Ex 3.17 An urn contains 10 white and 12 red chips. Two chips are drawn at random and, without looking at their colors, are discarded. What is the probability that a third chip drawn is red?

Sol 1: Conditioning on partition
$\{R_1R_2, R_1W_2, W_1R_2, W_1W_2\}$.

Sol 2: Think from a different physical stetting.

Bayes formula:

$$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(AB_j)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i)}$$

It is useful to think

- B_j, $1 \le j \le n$, as all possible "hypotheses".
- $\mathbb{P}(B_j)$, $1 \le j \le n$, as opinions about these "hypotheses" held before the experiment.
- The event A as the new evidence or outcomes
- $\mathbb{P}(B_j|A)$, $1 \le j \le n$, as new opinions based on the evidence of the experiment.

Ex 3.19: See textbook, p102.

Sol: Let G and I be the events that the suspect is guilty and innocent, respectively. Let D be the event that DNA matches. Then

$$\mathbb{P}(G|D) = ...$$

and $\mathbb{P}(I|D) = 1 - \mathbb{P}(G|D)$.

Ex 3.21: A box contains seven red and 13 blue balls. Two balls are selected at random and are discarded without their colors being seen. If a third ball is drawn randomly and observed to be red, what is the probability that both of the discarded balls were blue?

Sol 1: Let BB, BR, and RR be the events that the discarded balls are blue and blue, blue and red, red and red, respectively. Also, let R be the event that the third ball drawn is red. Then

$$\mathbb{P}(BB|R) = \frac{\mathbb{P}(BB \text{ and } R)}{\mathbb{P}(R)} = \frac{\mathbb{P}(R|BB)\mathbb{P}(BB)}{\text{using law of total prob.}}$$

Sol 2: Let B_1B_2, B_1R_2, R_1B_2 and R_1R_2 be the events that the first and second discarded balls are blue and blue, blue and red, red and blue, red and red, respectively. Also, let R_3 be the event that the third ball drawn is red. Then

$$\mathbb{P}(B_1B_2|R_3) = \frac{\mathbb{P}(B_1B_2R_3)}{\mathbb{P}(R_3)} = \frac{\mathbb{P}(R_3|B_1B_2)\mathbb{P}(B_1B_2)}{\text{using law of total prob.}}$$

Sol 3: Reduced sample space.

Independent Events

Two events A and B are *independent* if

$$\mathbb{P}(A|B) = \mathbb{P}(A), \quad \text{or} \quad \mathbb{P}(AB) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Ex: A card is selected at random from an ordinary deck of 52 playing cards. If A is the event that the selected card is a heart, B is the event that it is an ace and C is the event that it is a spade. Are A and B independent? Are B and C independent?

Ex : Toss a fair coin twice and define

$$\begin{aligned} A &= \{\text{first one is } H\}, \quad & B &= \{\text{second one is } H\}, \\ C &= \{\text{exactly one } H\}, \quad & D &= \{\text{second one is } T\}. \end{aligned}$$

Then A and B, A and C, A and D, B and C, C and D are independent. B and D are dependent events.

Ex: Toss two fair dice and define $A = \{\text{Sum} = 6\}$, $B = \{\text{Sum} = 7\}$ and $C = \{\text{the first is } 4\}$. Then A and C are dependent events but B and C are independent.

Theorem: If A and B are independent, then so are A and B^c.

Idea of proof: Write down what is given and what is asked. Make the difference disappear!

Corollary: If A and B are independent, then so are A^c and B^c.

Ex: Toss two fair dice and define $E = \{\text{Sum} = 7\}$, $F = \{\text{the first is } 4\}$ and $G = \{\text{the second is } 3\}$. Then E and F are independent, and E and G are independent. But E and FG are NOT independent.

Def: The events $A_1, \cdots, A_n$ are independent if for every subset $\{i_1, \cdots, i_r\} \subset \{1, 2, \cdots, n\}$, $r \leq n$,

$$\mathbb{P}(A_{i_1} A_{i_2} \cdots A_{i_r}) = \mathbb{P}(A_{i_1}) \mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_r}).$$

And an infinite set of events are independent if every finite of them are independent.

• This is typical way in mathematics to extend a definition from two to finite many, and then to infinite many.

Ex: For the coin toss example on the previous page, A, B and C are NOT independent.

Ex: Independent trials, consisting of rolling a pair of fair dice, are performed. What is the probability that an outcome of 5 appears before an outcome of 7 when the outcome of a roll is the sum of the dice? (see also Example 3.31 in the textbook).

Sol 1: (disjoint partition and approximate infinite by finite terms). Let

$$E_n = \{\text{no 5 or 7 on the first } n-1 \text{ trials and 5 on the } n\text{-th trials}\}$$

Then

$$\begin{aligned}\mathbb{P}(5 \text{ before } 7) &= \mathbb{P}(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mathbb{P}(E_n) \\ &= \sum_{n=1}^{\infty} (26/36)^{n-1}(4/36) = 2/5.\end{aligned}$$

Sol 2: (Reduced sample space). $\mathbb{P}(5) = 4/36$ and $\mathbb{P}(7) = 6/36$. Thus on the reduced sample space,

$$\mathbb{P}(5 \text{ before } 7) = \frac{\mathbb{P}(5)}{\mathbb{P}(5) + \mathbb{P}(7)} = 2/5$$

Sol 3: (The first step conditioning). Let

$$\begin{aligned} E &= \{5\,\text{before}\,7\} \\ F &= \{\text{the first trial is 5}\} \\ G &= \{\text{the first trial is 7}\} \\ H &= \{\text{the first trial is neither 5 nor 7}\}. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{P}(E) &= \mathbb{P}(E|F)\cdot\mathbb{P}(F)+\mathbb{P}(E|G)\cdot\mathbb{P}(G)+\mathbb{P}(E|H)\cdot\mathbb{P}(H) \\ &= 1\cdot\frac{4}{36}+0\cdot\frac{6}{36}+\mathbb{P}(E)\cdot\frac{26}{36}. \end{aligned}$$

Note the connection with sol. 2:

$$\begin{aligned} \mathbb{P}(E) &= \sum_{n=1}^{\infty}\frac{1}{9}(13/18)^{n-1} \\ &= 1/9+(13/18)\sum_{n=1}^{\infty}\frac{1}{9}(13/18)^{n-1} \\ &= 1/9+(13/18)\mathbb{P}(E) \end{aligned}$$

which is a way of finding geometric sums.

Honor's Section Only

- Applications of Probability to Genetics

- The Hardy-Weinberg Law

HW: (**Due March 3**) S3.6: **10**, **12**.