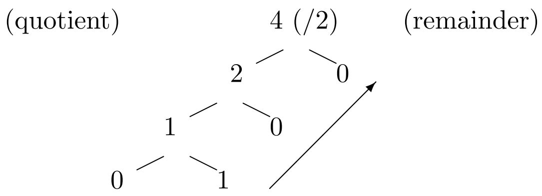
M353 Study Guide for the Final (S. Zhang) .

1. Convert $a = 4.125$ and $b = 19/7$ to IEEE doubles. Then find the IEEE double form for $a + b$ (using chopping after bit 52.)

   - **ans:**

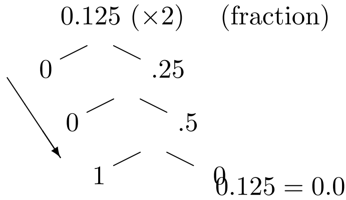
$$4.125 = 4 + 0.125$$

Convert 4



$$4 = 100_2$$

Convert 0.125.



$$0.125 = 0.001_2,$$

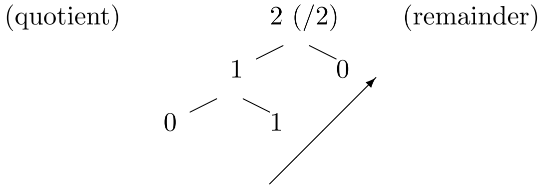
$$4.125 = 100.001_2 = 1.00001_2 \times 2^2$$

Use IEEE double: $p = 2$. $1111111111_2 + 2 = 1\,00000\,00001_2$

$$\boxed{0}\,\boxed{1\,00000\,00001}\,\underbrace{\boxed{000010...0}}_{52\text{ bit}}$$

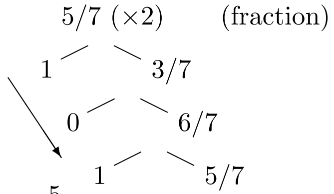
$$19/7 = 2 + \frac{5}{7}$$

Convert 2



$$2 = 10_2$$

Convert 5/7.



$$\frac{5}{7} = 0.\overline{101}_2$$

$$\frac{19}{7} = 10.\overline{101}_2 = 1.0\overline{101}_2 \times 2^1$$

Use IEEE double: $p = 1$. $1111111111_2 + 1 = 1\,00000\,00000_2$

$$\boxed{0}\,\boxed{1\,00000\,00000}\,\underbrace{\boxed{0\,101\,101\,...101}}_{52\text{ bit}}$$

$$\begin{aligned}
&4.125 + \frac{19}{7}\\
=&1.00001_2 \times 2^2 + 1.0\overline{101}_2 \times 2^1\\
=&1.\underbrace{\boxed{00\,001\,000\,...000\,00}}_{52\text{ bit}} \times 2^2\\
+&0.\underbrace{\boxed{10\,101\,101\,...101\,10}}_{52\text{ bit}} \times 2^2\\
=&1.\underbrace{\boxed{10\,110\,101\,...101\,10}}_{52\text{ bit}} \times 2^2
\end{aligned}$$

We can convert the answer back to decimal, but not required.

2. Find an interval of length one that contains a root.

$$x^4 = x^2 + 2$$

Then do two steps of bisection method. How many steps of bisection iterations are need to reach 10^{-10} accuracy?

   - **ans:**

$$f(x) = x^4 - x^2 - 2$$

$$f(0) = -2,\ f(1) = -2,\ f(2) = 10$$

$$[a, b] = [1, 2]$$

After we compute the first c, we will replace either a or b by c so that the new interval can still trap a root inside it.

$$c = \frac{a+b}{2}$$

i	$a, f(a)$	$c, f(c)$	$b, f(b)$
0	$1, [-2]$	$1.5, [0.8125]$	$2, [10]$
1	$1, [-2]$	$1.25, [-1.1211]$	$1.5, [0.8125]$
2	$1.25, [-1.1211]$		$1.5, [0.8125]$

$$x = 1.375$$

Error bound:

$$|r - c_n| \le \frac{1}{2^{n+1}}(b-a)$$

$$\frac{1}{2^{n+1}}(2-1) \le 10^{-10}$$

$$(n+1)\ln\frac{1}{2} \le \ln 10^{-10}$$

$$(n+1) \ge \frac{-10\ln 10}{\ln\frac{1}{2}} = 33.21$$

$$n \ge 33.$$

3. For finding the root of the function:

$$f(x) = x^2 - 4x - 12$$

   (a) Do 3 steps of the Newton's method, $p_0 = 3$. Find the errors and use the data to show the method is a second-order one. Then find the fixed point, and the convergence order and rate there.

   (b) Do 3 steps of the secant method, $p_0 = 2$, $p_1 = 8$.

   (c) Do 3 steps of the false position method, given the initial interval $[2, 8]$.

   (d) Construct a function with p_0 and p_1 ($p_0 < p_1$) so that p_3 in the secant method is different from the second iterate in the false position method, where the initial interval is $[p_0, p_1]$.

- **ans:**

(a)

$$f'(x) = 2x - 4$$

If $p_0 = 2$, then $f'(p_0) = 0$ and $p_1 = \infty$. The iteration diverges.

$$\begin{aligned}
p_0 &= 3, & f(p_0) &= -15\\
p_1 &= p_0 - f(p_0)/f'(p_0) = 10.5, & f(p_1) &= 56.25\\
p_2 &= p_1 - f(p_1)/f'(p_1) = 7.1911, & f(p_2) &= 10.95\\
p_3 &= 6.1366, & f(p_3) &= 1.112
\end{aligned}$$

From three iterations, it is hard to say the method is a second-order one, this is because the initial p_0 is too far away from the root. If we do more iterations:

$$\begin{aligned}
p_0 &= 3, & f(p_0) &= -15\\
p_1 &= p_0 - f(p_0)/f'(p_0) = 10.5, & f(p_1) &= 56.25\\
p_2 &= p_1 - f(p_1)/f'(p_1) = 7.1911, & f(p_2) &= 10.95\\
p_3 &= 6.1366, & f(p_3) &= 1.112\\
p_4 &= 6.0023, & f(p_4) &= 0.018\\
p_5 &= 6.00000064, & f(p_5) &= 0.0000051\\
p_6 &= 6.00000000000005
\end{aligned}$$

The root is 6. We can see we got the correct digits doubled, from 1, to 3, to 7, and to 14! We say the method is of second order.

(b) The secant method is almost the same as the method of false position, but simpler. We always use two new solutions to compute the next solution in the secant method. But in the method of false position, we use the two solutions at which the function has two different signs.

$$\begin{aligned}
p_0 &= 2, & f(p_0) &= -16\\
p_1 &= 8, & f(p_1) &= 20\\
p_2 &= p_1 - f(p_1)\frac{p_1 - p_0}{f(p_1) - f(p_0)}\\
&= 4.67, & f(p_2) &= -8.89\\
p_3 &= 5.69, & f(p_3) &= -2.36\\
p_4 &= 6.06, & f(p_4) &= 0.52
\end{aligned}$$

(c)

$$\begin{aligned}
f(2) &= -16, \qquad a = 3 = p_0\\
f(8) &= 20, \qquad b = 8 = p_1\\
c_0 = p_2 &= p_1 - f(p_1)\frac{p_1 - p_0}{f(p_1) - f(p_0)} = 4.67\\
f(c_0) &= f(p_2) = -8.89\\
c_1 = p_3 &= 5.69, \qquad f(p_3) = -2.36\\
p_4 \ne c_3 &= p_1 - f(p_1)\frac{p_1 - p_3}{f(p_1) - f(p_3)} = 5.96
\end{aligned}$$

Repeating above steps of the method of secant, until the 4th step, where the middle point is computed by p_1 and p_3 while the secant method uses p_2 and p_3.

k	$a_k(f(a))$	$c = b - \frac{f(b)(b-a)}{f(b)-f(a)}$	b_k
0	$p_0, 2_{-16}$	4.67−8.89	$p_1, 8_{+20}$
1	$p_2, 4.67{-}8.89$	5.69−2.36	$p_1, 8_{+20}$
2	$p_3, 5.69{-}2.36$	5.93(using p_1 and p_3)	$p_1, 8_{+20}$

Comparing the two methods, we can see the secant method is faster, but the secant method does not guarantee the convergence, while the method of false position does.

(d) From the table above in the false position method, we can see that to make c_1 different from p_3, c_0 must go the right column, instead of the left column, because this way, c_1 is computed by p_0 and p_2 while p_3 is by p_1 and p_2.

Let

$$\begin{aligned}
f(x) &= 4 - x^2 \\
a_0 &= p_0 = -3 \\
f(a_0) &= -5 \\
b_0 &= p_1 = -1 \\
f(b_0) &= 3 \\
c_0 &= p_2 = -1.75 \\
f(c_0) &= 0.93 > 0 \\
a_1 &= a_0, \quad b_1 = c_0, \qquad \text{(move to right!)} \\
p_3 &= p_2 - f(p_2)\frac{p_2 - p_1}{f(p_2) - f(p_1)} \\
&= -2.09, \qquad \text{(Secant Method)} \\
c_1 &= b_1 - f(b_1)\frac{b_1 - a_1}{f(b_2) - f(a_0)} \\
&= p_2 - f(p_2)\frac{p_2 - p_0}{f(p_2) - f(p_0)} \\
&= -1.94, \qquad \text{(False position)} \\
c_1 &\neq p_3.
\end{aligned}$$

4. Find convergence order and the rate of the Newton's method

$$32x^3 - 32x^2 - 6x + 9 = 0, \; r = -1/2, 3/4$$

- **ans:**

$$\begin{aligned}
f'(x) &= 96x^2 - 64x - 6 \\
f''(x) &= 192x - 64
\end{aligned}$$

$$\begin{aligned}
r &= -1/2 \\
f'(r) &= 50 \neq 0 \\
f''(r) &= -160 \\
\frac{e_{i+1}}{e_i^2} &\to M = \left|\frac{f''(r)}{2f'(r)}\right| = \frac{8}{5}
\end{aligned}$$

We have a quadratic convergence.

$$\begin{aligned}
r &= 3/4 \\
f'(r) &= 0 \\
f''(r) &= 80 \neq 0 \\
\frac{e_{i+1}}{e_i} &\to S = \frac{m-1}{m} = \frac{2-1}{2} = \frac{1}{2}
\end{aligned}$$

We have a linear convergence.

5. Solve the following system $(A|b)$

$$\begin{pmatrix} 1 & -3 & 3 & | & 4 \\ -2 & 0 & 1 & | & -1 \\ 3 & -1 & -1 & | & 2 \end{pmatrix}$$

(a) by GE without pivoting,

(b) by finding $A = LU$ and using it

(c) by GE with partial pivoting,

(d) by finding $PA = LU$ and using it

- **ans:**

(a) by GE without pivoting,

$$\begin{pmatrix} 1 & -3 & 3 & | & 4 \\ -2 & 0 & 1 & | & -1 \\ 3 & -1 & -1 & | & 2 \end{pmatrix}$$

$$\xrightarrow{2r_1+r_2} \begin{pmatrix} 1 & -3 & 3 & | & 4 \\ \boxed{-2} & -6 & 7 & | & 7 \\ 3 & -1 & -1 & | & 2 \end{pmatrix}$$

$$\xrightarrow{(-3)r_1+r_3} \begin{pmatrix} 1 & -3 & 3 & | & 4 \\ \boxed{-2} & -6 & 7 & | & 7 \\ \boxed{3} & 8 & -10 & -10 & \end{pmatrix}$$

$$\xrightarrow{(4/3)r_2+r_3} \begin{pmatrix} 1 & -3 & 3 & | & 4 \\ \boxed{-2} & -6 & 7 & | & 7 \\ \boxed{3} & \boxed{-4/3} & -2/3 & -2/3 & \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{(bottom up)}$$

(b) From last step in Gaussian elimination above, we get

$$A = LU = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ 3 & -4/3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 \\ & -6 & 7 \\ & & -2/3 \end{pmatrix}$$

(please check it.)

When solving $Ax = (LU)x = b$, we need to do two steps

$$\begin{aligned}
&Ly = b \\
&\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 3 & -4/3 & 1 \end{pmatrix} y == \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \\
&y = \begin{pmatrix} 4 \\ 7 \\ -2/3 \end{pmatrix} \qquad \text{(top down)} \\
&Ux = y \\
&\begin{pmatrix} 1 & -3 & 3 \\ & -6 & 7 \\ & & -2/3 \end{pmatrix} x = \begin{pmatrix} 4 \\ 7 \\ -2/3 \end{pmatrix} \\
&x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{(bottom up)}
\end{aligned}$$

(c) GE with partial pivoting (must switch rows exactly):

$$\left(\begin{array}{ccccc} 1 & -3 & 3 & | & 4 \\ -2 & 0 & 1 & | & -1 \\ 3 & -1 & -1 & | & 2 \end{array}\right)$$

$$\overset{r_1 \leftrightarrow r_3}{\longrightarrow} \left(\begin{array}{cccccc} 3 & -1 & -1 & | & 2 & r_3 \\ -2 & 0 & 1 & | & -1 & r_2 \\ 1 & -3 & 3 & | & 4 & r_1 \end{array}\right)$$

$$\overset{(2/3)r_1 + r_2}{\longrightarrow} \left(\begin{array}{cccccc} 3 & -1 & -1 & | & 2 & r_3 \\ \boxed{-2/3} & -2/3 & 1/3 & | & 1/3 & r_2 \\ 1 & -3 & 3 & | & 4 & r_1 \end{array}\right)$$

$$\overset{(-1/3)r_1 + r_3}{\longrightarrow} \left(\begin{array}{cccccc} 3 & -1 & -1 & | & 2 & r_3 \\ \boxed{-2/3} & -2/3 & 1/3 & | & 1/3 & r_2 \\ \boxed{1/3} & -8/3 & 10/3 & | & 10/3 & r_1 \end{array}\right)$$

$$\overset{r_2 \leftrightarrow r_3}{\longrightarrow} \left(\begin{array}{cccccc} 3 & -1 & -1 & | & 2 & r_3 \\ \boxed{1/3} & -8/3 & 10/3 & | & 10/3 & r_1 \\ \boxed{-2/3} & -2/3 & 1/3 & | & 1/3 & r_2 \end{array}\right)$$

$$\overset{(-1/4)r_2 + r_3}{\longrightarrow} \left(\begin{array}{cccccc} 3 & -1 & -1 & | & 2 & r_3 \\ \boxed{1/3} & -8/3 & 10/3 & | & 10/3 & r_1 \\ \boxed{-2/3} & \boxed{1/4} & -1/2 & | & -1/2 & r_2 \end{array}\right)$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (\text{ bottom up })$$

(d) by finding $PA = LU$ and using it
From above work, we have

$$P = \begin{pmatrix} 0 & 0 & 1 & r_3 \\ 1 & 0 & 0 & r_1 \\ 0 & 1 & 0 & r_2 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & \\ 1/3 & 1 & \\ -2/3 & 1/41 & \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & -1 & -1 \\ & -8/3 & 10/3 \\ & & -1/2 \end{pmatrix}$$

$$PA = LU$$

(check it.) Now, we have three steps:

$$z = Pb \qquad \Rightarrow z = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

$$Ly = z \qquad \Rightarrow y = \begin{pmatrix} 2 \\ 10/3 \\ -1/2 \end{pmatrix}$$

$$Ux = y \qquad \Rightarrow x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

6. Let

$$A = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Find x_2 if Jacobi iteration is used.

(b) Find the exact solution x and errors for the above Jacobi iteration, $\|x - x_i\|_\infty$, $i = 0, 1, 2$.

(c) Find the error reduction bound for the Jacobin iteration, $\|R_j\|_\infty$. Check the error reduction data above.

(d) Find x_2 if Gauss–Seidel iteration is used.

(e) Find the error reduction bound for the Gauss–Seidel iteration, $\|R_{gs}\|_\infty$.

- **ans:**

(a) We use the notation for splitting a matrix to a strictly lower-triangular part, a diagonal part, and a strictly upper-triangular part:

$$A = L_t + D + U_t$$

$$\begin{aligned} x_1 &= D^{-1}b - D^{-1}(L_t + U_t)x_0 \\ &= \begin{pmatrix} 0.25 & 0. & 0. \\ 0. & 0.25 & 0. \\ 0. & 0. & 0.25 \end{pmatrix} \begin{pmatrix} 2. \\ 2. \\ 2. \end{pmatrix} \\ &- \begin{pmatrix} 0.25 & 0. & 0. \\ 0. & 0.25 & 0. \\ 0. & 0. & 0.25 \end{pmatrix} \begin{pmatrix} 0. & -1. & -1. \\ -1. & 0. & -1. \\ -1. & -1. & 0. \end{pmatrix} \begin{pmatrix} 1. \\ 0. \\ 1. \end{pmatrix} \\ &= \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} - \begin{pmatrix} 0. & -0.25 & -0.25 \\ -0.25 & 0. & -0.25 \\ -0.25 & -0.25 & 0. \end{pmatrix} \begin{pmatrix} 1. \\ 0. \\ 1. \end{pmatrix} \\ &= \begin{pmatrix} 0.75 \\ 1. \\ 0.75 \end{pmatrix} \end{aligned}$$

Repeat one more time.

$$\begin{aligned} x_2 &= D^{-1}b - D^{-1}(L_t + U_t)x_1 \\ &= \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} - \begin{pmatrix} 0. & -0.25 & -0.25 \\ -0.25 & 0. & -0.25 \\ -0.25 & -0.25 & 0. \end{pmatrix} \begin{pmatrix} 0.75 \\ 1. \\ 0.75 \end{pmatrix} \\ &= \begin{pmatrix} 0.9375 \\ 0.875 \\ 0.9375 \end{pmatrix} \end{aligned}$$

The iteration converges to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(b) We can use Gaussian elimination or simply checking the observation above to get the exact solution

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So we have errors

$$\|e_0\|_\infty = \|x - x_0\|_\infty = \| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \|_\infty = 1$$

$$\|e_1\|_\infty = \|x - x_1\|_\infty = \| \begin{pmatrix} .25 \\ 0 \\ .25 \end{pmatrix} \|_\infty = .25$$

$$\|e_2\|_\infty = \|x - x_2\|_\infty = \| \begin{pmatrix} .0625 \\ .125 \\ .0625 \end{pmatrix} \|_\infty = .125$$

(c)

$$\begin{aligned} R_j &= -D^{-1}(L_t + U_t) \\ &= \begin{pmatrix} 0. & .25 & .25 \\ .25 & 0. & .25 \\ .25 & .25 & 0. \end{pmatrix} \end{aligned}$$

$$\|R_j\|_\infty = \max\{0 + .25 + .25, .5, .5\} = \frac{1}{2}.$$

Checking the reduction is no less than 1/2:

$$\frac{\|e_1\|_\infty}{\|e_0\|_\infty} = \frac{.25}{1} = \frac{1}{4} < \frac{1}{2}$$

$$\frac{\|e_2\|_\infty}{\|e_1\|_\infty} = \frac{.125}{.25} = \frac{1}{2} \leq \frac{1}{2}!$$

(d)

$$\begin{aligned} (L_t + D)^{-1} &= \begin{pmatrix} 4. & 0. & 0. \\ -1. & 4. & 0. \\ -1. & -1. & 4. \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0.25 & 0. & 0. \\ 0.0625 & 0.25 & 0. \\ 0.078125 & 0.0625 & 0.25 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} x_1 &= (L_t + D)^{-1}b - (L_t + D)^{-1}U_t x_0 \\ &= \begin{pmatrix} 0.25 & 0. & 0. \\ 0.0625 & 0.25 & 0. \\ 0.078125 & 0.0625 & 0.25 \end{pmatrix} \begin{pmatrix} 2. \\ 2. \\ 2. \end{pmatrix} \\ &- \begin{pmatrix} 0.25 & 0. & 0. \\ 0.0625 & 0.25 & 0. \\ 0.078125 & 0.0625 & 0.25 \end{pmatrix} \begin{pmatrix} 0. & -1. & -1. \\ 0. & 0. & -1. \\ 0. & 0. & 0. \end{pmatrix} \begin{pmatrix} 1. \\ 0. \\ 1. \end{pmatrix} \\ &= \begin{pmatrix} 0.5 \\ 0.625 \\ 0.78125 \end{pmatrix} - \begin{pmatrix} 0. & -0.25 & -0.25 \\ 0. & -0.0625 & -0.3125 \\ 0. & -0.078125 & -0.140625 \end{pmatrix} \begin{pmatrix} 1. \\ 0. \\ 1. \end{pmatrix} \\ &= \begin{pmatrix} 0.75 \\ 0.9375 \\ 0.921875 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} x_2 &= (L_t + D)^{-1}b - (L_t + D)^{-1}U_t x_1 \\ &= \begin{pmatrix} 0.5 \\ 0.625 \\ 0.78125 \end{pmatrix} - \begin{pmatrix} 0. & -0.25 & -0.25 \\ 0. & -0.0625 & -0.3125 \\ 0. & -0.078125 & -0.140625 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 0.75 \\ 0.9375 \\ 0.921875 \end{pmatrix} = \begin{pmatrix} 0.96484375 \\ 0.971679688 \\ 0.984130859 \end{pmatrix}$$

The solution is much better than that of the Jacobi iteration.

(e)

$$\begin{aligned} R_{gs} &= -(L_t + D)^{-1}U_t \\ &= \begin{pmatrix} 0. & -0.25 & -0.25 \\ 0. & -0.0625 & -0.3125 \\ 0. & -0.078125 & -0.140625 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \|R_{gs}\|_\infty = \max\{&0.25 + 0.25, 0.0625 + 0.3125, \\ &0.078125 + 0.140625\} = \frac{1}{2}. \end{aligned}$$

7. Find the $P_3(x)$ interpolation by
   (1) solving equations for unknown coefficients,
   (2) Lagrange nodal basis,
   (3) Newton's divided differences.

x_i	-1	0	1	2
y_i	0	-2	0	0

- **ans:** Method(1) We look for a degree 3 polynomial to fit 4 data points.

$$P_3(x) = a + bx + cx^2 + dx^3$$

For example, when $x_0 = -1$, $y_0 = 0$ we get one equation:

$$0 = a - b + c - d$$

Then for the unknown coefficients (vector $X = (a, b, ...)$), we get the following equations obtained by evaluating the polynomial at the given points:

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 0. \end{pmatrix}$$

Solve the system of equations:

$$X = \begin{pmatrix} -2 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

$$P_3(x) = -2 + x + 2x^2 - x^3$$

Method(2) Using Lagrange basis

$$\begin{aligned}
p_3(x) =& y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \\
&+ y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\
&+ y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \\
&+ y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\
=& 0 - 2\frac{(x+1)(x-1)(x-2)}{(1)(-1)(-2)} + 0 + 0 \\
=& -(x^2-1)(x-2) \\
=& -x^3 + 2x^2 + x - 2
\end{aligned}$$

We must get the same answer. Also we can check the function by evaluate it at the 4 given point!

Method (3) Newton's divided differences:

$$\begin{aligned}
f[x_k] &= f(x_k) \\
f[x_k, x_{k+1}] &= \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k} \\
f[x_k, x_{k+1}, x_{k+2}] &= \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k}
\end{aligned}$$

$$\begin{aligned}
& f[x_k, x_{k+1}, x_{k+2}, x_{k+3}]] \\
=& \frac{f[x_{k+1}, x_{k+2}, x_{k+3}] - f[x_k, x_{k+1}, x_{k+2}]}{x_{k+3} - x_k}
\end{aligned}$$

$$P = \sum_{i=0}^{n} f[x_0 ... x_i](x - x_0)...(x - x_{i-1})$$

We can use the above formulas directly. But it is much easier to use the following table.

$$\left(\begin{array}{ccccc}
-1 & \boxed{0} & & & \\
 & & \boxed{-2} & & \\
0 & -2 & & \boxed{2} & \\
 & & 2 & & \boxed{-1} \\
1 & 0 & & -1 & \\
 & & 0 & & \\
2 & 0 & & &
\end{array}\right)$$

$$\begin{aligned}
P_3 &= 0 - 2(x+1) + 2(x+1)(x) - (x+1)x(x-1) \\
&= -2x - 2 + 2x^2 + 2x - x^3 + x \\
&= -x^3 + 2x^2 + x - 2
\end{aligned}$$

8. Find the least-squares solution each problem.

$$\begin{pmatrix} -2 & -1 \\ 1 & 1 \\ 3 & -1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 0 \\ 14 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 & 3 \\ -1 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 21 \\ 9 \end{pmatrix}.$$

- **ans:**

$$A^T A x = A^T b$$

$$A^T A = \begin{pmatrix} 14 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 28 \\ -21 \end{pmatrix}$$

$$x = (A^T A)^{-1} A^T b = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$$

$$AA^T y = b \Rightarrow y = \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$$

$$x = A^T y = \begin{pmatrix} -6 \\ 4.5 \\ 1.5 \end{pmatrix}$$

9. Let $f(x) = x + \sin x$, $x_0 = 0.5$.

   (a) Approximate $f'(x_0)$ by the (3-point) central differences with $h = 0.1$ and $h = 0.2$. Find the error bound for $h = 0.1$. Find the Richardson extrapolation. Check all three errors.

   (b) Using the (3-point) central differences with $h = 0.1$ and $h = 0.2$, Find the error bound for $h = 0.1$. Find the extrapolation to approximate $f''(x_0)$. Check all three errors.

- **ans:**

(a)

$$\begin{aligned}
f_0' &= \frac{f_1 - f_{-1}}{2h} \\
f_0' &= \frac{f(x_0 + h) - f(x_0 - h)}{2h} \\
&= 1.8761 \\
e = f'(x_0) - f_0' &= 0.0015
\end{aligned}$$

Error bound:

$$f''' = -\cos(x)$$

$$\begin{aligned}|f'(x_0) - f'_0| &\leq \frac{h^2}{6} \max x_{-1} \leq z \leq x_1 |f'''(z)| \\ &\leq \frac{0.1^2}{6}(\cos(0.5-0.1)) = 0.0015\end{aligned}$$

$$\begin{aligned}f'_0(h=0.1) &= 1.8761 \\ f'_0(h=0.2) &= 1.8717 \\ e(h=0.2) &= 0.0058 \\ f'_0 &= \frac{4f'_0(h=0.1) - f'_0(h=0.2)}{4-1} \\ &= 1.8776 \\ \text{new error} = f'(x_0) - f'_0 &= 0.0000029\end{aligned}$$

The new error must be much smaller!

(b)

$$\begin{aligned}f''_0 &= \frac{f_1 - 2f_0 + f_{-1}}{h^2} \\ f''_0(h=0.1) &= \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} \\ &= -0.4790 \\ f''(x_0) - f''_0 &= -0.0004\end{aligned}$$

Error bound:

$$f'''' = \sin(x)$$

$$\begin{aligned}|f'(x_0) - f'_0| &\leq \frac{h^2}{12} \max x_{-1} \leq z \leq x_1 |f''''(z)| \\ &\leq \frac{0.1^2}{12}(sin(0.5+0.1)) = 0.00047\end{aligned}$$

$$\begin{aligned}f''_0(h=0.2) &= -0.4778 \\ f''(x_0) - f''_0 &= -0.0016 \\ f''_0 &= \frac{4f''_0(h=0.1) - f''_0(h=0.2)}{4-1} \\ &= -0.4794 \\ f''(x_0) - f''_0 &= -0.00000053\end{aligned}$$

10. Compute $\int_0^2 5x^4\,dx$.
    (a) Use the trapezoidal rule with $m = 1 (h = 2)$, and $m = 2$. Find the extrapolation. Find all three errors.
    (b) Use the mid-point rule with $m = 1 (h = 2)$, and $m = 2$. Find the extrapolation. Find all three errors.
    (c) Use the Simpson's rule with $m = 1 (h = 1)$, and $m = 2$. Find the extrapolation. Find all three errors.

- **ans:**

Exact value

$$\int_0^2 5x^4\,dx = 32$$

(a) Trapezoidal rule:
$m = 1$, $h = 2$:

$$\begin{aligned}T_h &= \frac{h}{2}(f_0 + f_1) \\ &= \frac{h}{2}(f(0) + f(2)) \\ &= 80 \\ err &= -48\end{aligned}$$

$m = 2$, $h = 1$:

$$\begin{aligned}T_h &= \frac{h}{2}(f_0 + 2f_1 + f_2) \\ &= \frac{h}{2}(f(0) + 2f(1) + f(2)) \\ &= 45 \\ err &= -13\end{aligned}$$

Extrapolation (second order)

$$\begin{aligned}extrap &= \frac{2^2 T_{m=2} - T_{m=1}}{2^2 - 1} \\ &= 33.333333 \\ err &= -1.3333333\end{aligned}$$

(b) Midpoint rule:
$m = 1$, $h = 2$:

$$\begin{aligned}M_h &= h(f_{1/2}) = \frac{h}{2}(f(1)) \\ &= 10 \\ err &= 22\end{aligned}$$

$m = 2$, $h = 1$:

$$\begin{aligned}M_h &= h(f_{1/2} + f_{3/2}) \\ &= h(f(0.5) + f(1.5)) \\ &= 25.625 \\ err &= 6.375\end{aligned}$$

Extrapolation (second order)

$$\begin{aligned}extrap &= \frac{2^2 M_{m=2} - M_{m=1}}{2^2 - 1} \\ &= 30.8333333 \\ err &= 1.1666666\end{aligned}$$

(c) Simpson's rule: $h = 1$:

$$\begin{aligned}S_h &= \frac{h}{3}(f(0) + 4f(1) + f(2)) \\ &= 33.33333333 \\ err &= -1.33333\end{aligned}$$

$h = 1/2$:

$$\begin{aligned}S_h &= \frac{h}{3}(f(0) + 4f(\frac{1}{2}) + 2f(1) + 4f(\frac{3}{2}) + f(2)) \\ &= 32.083333333 \\ err &= -0.08333333\end{aligned}$$

Extrapolation (4th order)

$$\begin{aligned} extrap &= \frac{2^4 S_{m=2} - S_{m=1}}{2^4 - 1} \\ &= 32 \\ err &= 0 \end{aligned}$$

11. Find Romberg R_{33} and the error for

$$\int_1^3 6x^5\, dx.$$

- **ans:**

$$\begin{aligned} R_{11} = T_{m=1} &= h[\frac{f_0}{2} + \frac{f_1}{2}] \\ &= 2[\frac{6(1^5)}{2} + \frac{6(3^5)}{2}] = 1464 \\ R_{21} = T_{m=2} &= \frac{R_{11}}{2} + hf_1 \\ &= \frac{R_{11}}{2} + 1(6(2^5)) = 934 \\ R_{31} = T_{m=4} &= \frac{R_{21}}{2} + h[f_1 + f_3] \\ &= 777.75 \\ R_{22} &= \frac{4R_{21} - R_{11}}{3} = 744 \\ R_{32} &= \frac{4R_{31} - R_{21}}{3} = 729 \\ R_{33} &= \frac{16R_{32} - R_{22}}{16 - 1} = 728 \end{aligned}$$

Exact

$$\int_1^3 6x^5\, dx = 3^6 - 1^6 = 728$$

Error is 0.

12. Compute Gauss integration G_2, G_3 ($x_i = \pm\sqrt{.6}, 0$, $c_i = \frac{5}{9}, \frac{8}{9}$) for

$$\int_1^2 x^4 + 1\, dx.$$

- **ans:** Formula (change variables – must apply Gauss formula on $[-1, 1]$)

$$\int_a^b f(z)dz = \int_{-1}^1 f(\frac{a+b}{2} + u\frac{b-a}{2})\frac{b-a}{2}du$$

$$\int_{-1}^1 \left(\left(\frac{u+3}{2}\right)^4 + 1\right)\frac{du}{2}$$

$$\begin{aligned} G_2 &= \frac{\left(\frac{-\frac{1}{\sqrt{3}}+3}{2}\right)^4 + 1}{2} + \frac{\left(\frac{\frac{1}{\sqrt{3}}+3}{2}\right)^4 + 1}{2} \\ &= 7.19444418 \end{aligned}$$

$$\begin{aligned} G_3 &= \frac{5}{9}\frac{\left(\frac{-\sqrt{.6}+3}{2}\right)^4 + 1}{2} + \frac{8}{9}\frac{\left(\frac{3}{2}\right)^4 + 1}{2} \\ &+ \frac{5}{9}\frac{\left(\frac{\sqrt{.6}+3}{2}\right)^4 + 1}{2} \\ &= 7.19999988 \end{aligned}$$

The exact integral is 7.2.

13. Find the exact solution and $y(1)$.
(1) Apply Euler method with $h = 1/2$ and $h = 1/4$ for $y(1)$ and the extrapolation, find the three errors.
(2) Apply backward Euler with $h = 1$ and $h = 1/4$ for $y(1)$ and the extrapolation, find the three errors.

$$y' = t - y,\ y(0) = 0$$

- **ans:**

1st order linear: $y' + py = q$.

$$\mu = e^{\int p} = e^t$$

sol:

$$\begin{aligned} y &= \mu^{-1}\int \mu q = e^{-t}\int e^t t dt \\ &= e^{-t}(e^t t - e^t + C) = t - 1 + Ce^{-t} \end{aligned}$$

$$y(0) = 0\ \Rightarrow\ y = t - 1 + e^{-t}$$

$$y(1) = e^{-1} = 0.3678794411714$$

Backward Euler: $y' = f(t, y)$.

$$\begin{aligned} y_1 &= y_0 + hf_1 = y_0 + hf(t_1, y_1) \\ &= y_0 + ht_1 - hy_1 \end{aligned}$$

$$y_1 = \frac{y_0 + ht_1}{1 + h}$$

$$y_{i+1} = \frac{y_i + ht_{i+1}}{1 + h}$$

$$h = 1/4, ;\ y_0 = 0,\ t_0 = 0$$

$$\begin{aligned} t_1 = 0.25;\quad y_1 &= \frac{0 + 0.25(0.25)}{1.25} \\ &= 0.05 \\ err &= y(t_i) - y_i = -0.0212 \end{aligned}$$

$$\begin{aligned} i = 2;\ t_i &= 0.5 \\ y_i &= 0.14 \\ err &= y(t_i) - y_i = -0.0335 \end{aligned}$$

$$\begin{aligned} i = 3;\ \ t_i &= 0.75 \\ y_i &= 0.262 \\ err = y(t_i) - y_i &= -0.0396 \end{aligned}$$

$$\begin{aligned} i = 4;\ \ t_i &= 1 \\ y_i &= 0.4096 \\ err = y(t_i) - y_i &= -0.0417 \end{aligned}$$

14. Solve the boundary value problem (find approximate values $x(1)$) :

$$x'' - x' = 4 - 4t, \qquad x(0) = 1, \quad x(2) = 9$$

(1) By linear Euler shooting with $h = 1$
(2) By nonlinear Euler shooting with $h = 1$, and 2 bisections starting with $x'(0) \in [0, 2]$ (4 shootings).
(3) By the finite difference method with $h = 1$.
The exact solution is $x = 2t^2 + 1$ (no use other than checking)

- **ans:**

Exact solution is

$$x(1) = 3$$

Shooting 1 – For u: $h = 1$,

$$U = Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$U' = F(t, U) = \begin{pmatrix} y \\ y + 4 - 4t \end{pmatrix}$$

$$\begin{aligned} U_1 &= U_0 + hF(t_0, U_0) \\ &= U_0 + h\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} U_2 &= U_1 + hF(t_1, U_1) \\ &= U_1 + h\begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix} \end{aligned}$$

Shooting 2 – For v:

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V' = F(t, V) = \begin{pmatrix} y \\ y \end{pmatrix}$$

$$\begin{aligned} V_1 &= V_0 + hF(t_0, V_0) \\ &= V_0 + h\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} V_2 &= V_1 + hF(t_1, V_1) \\ &= V_1 + h\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{aligned}$$

Combine them:

$$\begin{aligned} x(t) &= u + \frac{\beta - u(b)}{v(b)}v \\ &= u + \frac{9 - 5}{3}v \\ &= u + \frac{4}{3}v \end{aligned}$$

In particular,

$$\begin{aligned} x(1) &\sim u(1) + \frac{4}{3}v(1) = (U_1)_1 + \frac{4}{3}(V_1)_1 \\ &= \frac{7}{3} \end{aligned}$$

Error is 2/3.

Nonlinear shooting:

Nonlinear Shooting 1 – $x'(0) = 0$: $h = 1$,

$$Z = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \ Z_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Z' = F(t, Z) = \begin{pmatrix} y \\ y + 4 - 4t \end{pmatrix}$$

$$\begin{aligned} Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= Z_0 + h\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= Z_1 + h\begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix} \end{aligned}$$

$x(2) \simeq 5$ too low.

Nonlinear Shooting 2 – $x'(0) = 2$: $h = 1$,

$$Z = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \ Z_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$Z' = F(t, Z) = \begin{pmatrix} y \\ y + 4 - 4t \end{pmatrix}$$

$$\begin{aligned} Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= Z_0 + h\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= Z_1 + h\begin{pmatrix} 8 \\ 8 \end{pmatrix} = \begin{pmatrix} 11 \\ 16 \end{pmatrix} \end{aligned}$$

$x(2) \simeq 11$ too high.

<u>Nonlinear Shooting 3 – $x'(0) = 1$:</u> $h = 1$,

$$Z = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \ Z_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Z' = F(t, Z) = \begin{pmatrix} y \\ y + 4 - 4t \end{pmatrix}$$

$$\begin{aligned} Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= Z_0 + h\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= Z_1 + h\begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix} \end{aligned}$$

$x(2) \simeq 8$ too low. <u>Nonlinear Shooting 4 – $x'(0) = 1.5$:</u> $h = 1$,

$$Z = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \ Z_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Z' = F(t, Z) = \begin{pmatrix} y \\ y + 4 - 4t \end{pmatrix}$$

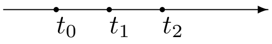
$$\begin{aligned} Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= Z_0 + h\begin{pmatrix} 1.5 \\ 5.5 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 7 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= Z_1 + h\begin{pmatrix} 7 \\ 7 \end{pmatrix} = \begin{pmatrix} 9.5 \\ 14 \end{pmatrix} \end{aligned}$$

$x(2) \simeq 9.5$ too high. If we bisect again, we try $x'(0) = 1.25$. So

$$x(1) \simeq Z_{1,1} = 2.5$$

<u>Finite difference for $h = 1$:</u>



When $h = 1$, only 1 unknown x_1. x_0 and x_2 and known.

$$\begin{aligned} &\frac{x_0 - 2x_1 + x_2}{h^2} - \frac{x_2 - x_0}{2h} = 4 - 4t_1 \\ &(-2)x_1 = -6 \\ &x_1 = 3 \end{aligned}$$

The error is 0!

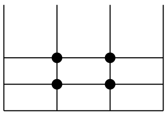
15. Solving the heat equation by the finite difference method. $u_t = 4u_{xx}$ for $t \in (0, 0.1)$ and $x \in (0, 3)$ with initial and boundary values

$$u(x, 0) = 2(3 - x)x, \quad u(0, t) = 0, \quad u(3, t) = 0.$$

Let $\Delta t = 0.1$. Find $u(x_i, 0.1)$ by the following methods with $h = 1$.

(a) the Euler (explicit) method.
(b) the backward Euler (implicit) method.
(c) the Crank-Nicholson method.

- **<u>ans:</u>**

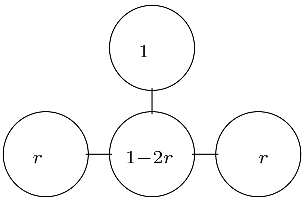
We cut the domain this way : 

For all three methods

$$r = c^2\frac{k}{h^2} = c^2\frac{\Delta t}{\Delta x^2} = 0.4$$

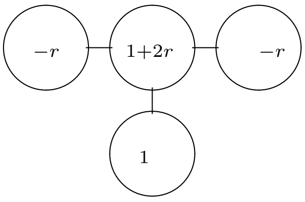
$$U_0 = \begin{pmatrix} f(x_1) \\ f(x_2) \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

<u>Euler method</u>:



$$U_1 = AU_0 = \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.2 \end{pmatrix} U_0 = \begin{pmatrix} 2.4 \\ 2.4 \end{pmatrix}$$

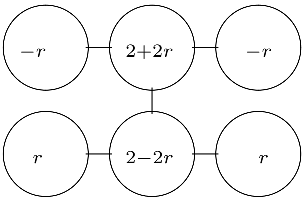
<u>Backward Euler</u>:



$$AU_1 = U_0, \ A = \begin{pmatrix} 1.8 & -0.4 \\ -0.4 & 1.8 \end{pmatrix}$$

$$\begin{aligned} U_1 &= A^{-1}U_0 \\ &= \begin{pmatrix} 0.58442 & 0.12987 \\ 0.12987 & 0.58442 \end{pmatrix} U_0 \\ &= \begin{pmatrix} 2.8571 \\ 2.8571 \end{pmatrix} \end{aligned}$$

<u>Crank-Nicholson</u>:



$$\begin{aligned} AU_1 &= BU_0 \\ A &= \begin{pmatrix} 2.8 & -0.4 \\ -0.4 & 2.8 \end{pmatrix}, \text{(see the graph)} \\ B &= \begin{pmatrix} 1.2 & 0.4 \\ 0.4 & 1.2 \end{pmatrix}, \\ A^{-1}B &= \begin{pmatrix} 0.45833 & 0.20833 \\ 0.20833 & 0.45833 \end{pmatrix} \\ U_1 = A^{-1}BU_0 &= \begin{pmatrix} 2.6667 \\ 2.6667 \end{pmatrix} \end{aligned}$$

16. Solving the wave equation by the finite difference method.

$$u_{tt} = 4u_{xx}, \quad t \in (0, .4),\ x \in (0, 1)$$

with initial and boundary values

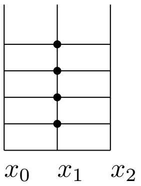
$$u(x, 0) = 4x(1 - x), \quad u_t(x, 0) = -x,$$

$$u(0, t) = 0, \quad u(1, t) = 0.$$

(a) Let $h = \Delta x = \frac{1}{2}$, $k = \Delta t = 0.1$. Find $u(x_i, 0.4)$.
(b) Let $h = \Delta x = \frac{1}{4}$, $k = \Delta t = 0.1$. Find $u(x_i, 0.2)$.

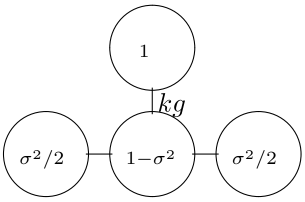
- **ans:**

(a) When $h = 1/2$, the grid is

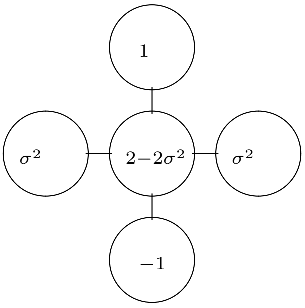


$$c = 2,\ \sigma = c\frac{k}{h} = c\frac{\Delta t}{\Delta x} = .4.$$

The graph for the first level is



The graph for high levels:



The grid points in x direction:

$$x_i = 0,\ \frac{1}{2},\ 1$$

For $u(x, 0) = 4x(1 - x)$, we get

$$\begin{aligned} \hat{U}_0 &= \begin{pmatrix} u(x_0, 0) \\ u(x_1, 0) \\ u(x_2, 0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ U_0 &= \left(u(\tfrac{1}{2}, 0)\right) = \left(1\right) \end{aligned}$$

For $u_t(x, 0) = -x$, we get

$$G = (U_t)_0 = \left(u_t(x_1, 0)\right) = \left(-1/2\right)$$

$$\begin{aligned} A &= tridiag\left(\sigma^2 \quad 2 - 2\sigma^2 \quad \sigma^2\right) \\ &= tridiag\left(.16 \quad 1.68 \quad .16\right) = \left(1.68\right) \end{aligned}$$

$$\begin{aligned} U_1 &= \frac{1}{2}AU_0 + kG \\ &= \frac{1}{2}\left(1.68\right)\left(1\right) + 0.1\left(-1/2\right) \\ &= \left(.79\right) \end{aligned}$$

Put the two boundary conditions on U_1, we have

$$\begin{aligned} \hat{U}_1 &= \begin{pmatrix} u(0, t_1) \\ \tilde{U}_1 \\ u(1, t_1) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ .79 \\ 0 \end{pmatrix} \end{aligned}$$

(No need to show $\hat{U}_1$, as we have 0 b.c. in this class.) The formula for higher levels are

$$\begin{aligned} \tilde{U}_2 &= AU_1 - U_0 \\ &= \left(1.68\right)U_1 - U_0 \\ &= \left(1.68\right)\left(.79\right) - \left(1\right) \\ &= \left(.3272\right) \end{aligned}$$

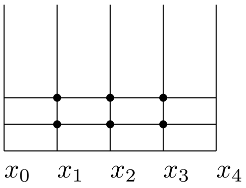
Now, repeating two more times, we get

$$U_3 = AU_2 - U_1 = \left(-.2403\right)$$

$$U_4 = AU_3 - U_2 = \left(-.7309\right)$$

We can draw the graph to see a string wave.

(b) For $h = 1/4$ we have



$$c = 2,\ \sigma = c\frac{k}{h} = c\frac{\Delta t}{\Delta x} = .8.$$

For $u(x,0) = 4x(1-x)$, we get

$$\hat{U}_0 = \begin{pmatrix} u(x_0,0) \\ u(x_1,0) \\ u(x_2,0) \\ u(x_3,0) \\ u(x_4,0) \end{pmatrix} = \begin{pmatrix} 0 \\ .75 \\ 1 \\ .75 \\ 0 \end{pmatrix}$$

$$U_0 = \begin{pmatrix} .75 \\ 1 \\ .75 \end{pmatrix}$$

For $u_t(x,0) = -x$, we get

$$G = (U_t)_0 = \begin{pmatrix} u_t(x_1,0) \\ u_t(x_2,0) \\ u_t(x_3,0) \end{pmatrix} = \begin{pmatrix} -.25 \\ -.5 \\ -.75 \end{pmatrix}$$

$$\begin{aligned} A &= tridiag\left(\sigma^2 \quad 2-2\sigma^2 \quad \sigma^2\right) \\ &= \begin{pmatrix} 0.72 & .64 & \\ .64 & 0.72 & .64 \\ & .64 & 0.72 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \tilde{U}_1 &= \frac{1}{2}AU_0 + kG \\ &= \frac{1}{2}A\begin{pmatrix} .75 \\ 1 \\ .75 \end{pmatrix} + 0.1\begin{pmatrix} -.25 \\ -.5 \\ -.75 \end{pmatrix} \\ &= \begin{pmatrix} .565 \\ .79 \\ .515 \end{pmatrix} \end{aligned}$$

The formula for higher levels are

$$\begin{aligned} U_2 &= AU_1 - U_0 \\ &= A\begin{pmatrix} .565 \\ .79 \\ .515 \end{pmatrix} - \begin{pmatrix} .75 \\ 1 \\ .75 \end{pmatrix} \\ &= \begin{pmatrix} .1624 \\ .26 \\ .1264 \end{pmatrix} \end{aligned}$$

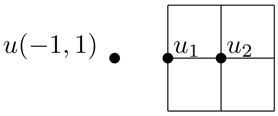
17. Solve the finite difference equations for the Laplace equation with $h = 1$.

$$u_{xx} + u_{yy} = 2 + 2x,\ 0 \le x, y \le 2$$

with the boundary condition

$$\begin{aligned} u_x(0,y) &= y^2, & u(2,y) &= 2y^2 + 4 \\ u(x,0) &= x^2, & u(x,2) &= 4x + x^2. \end{aligned}$$

• **ans:** The rectangle is cut by a vertical line $x = 1$ and a horizontal line $y = 1$. So we have 2 unknowns shown in the picture.



From the derivatives, the ghost point values are obtained by the central difference method

$$\frac{u_2 - u(-1,1)}{2h} = u_x(0,1) = y_1^2 = 1$$

$$u(-1,1) = u_2 - 2$$

At the unknown u points, we get equations

$$\begin{aligned} (u_2 - 2) + (x_0^2) - 4u_1 + u_2 + (4x_0 + x_0^2) &= (2 + 2x_0)h^2 \\ u_1 + (x_1^2) - 4u_2 + (4x_1 + x_1^2) + (2y_1^2 + 4) &= (2 + 2x_1)h^2 \end{aligned}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} u = \begin{pmatrix} 2+2-0 \\ 4-1-6-5 \end{pmatrix}$$

Solution

$$u = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

18. Approximate the area bounded by

$$4(x - \frac{1}{2})^2 \le y \le (1 - x^3)$$

by the Monte Carlo method with N=6 and the linear congruential generator:

$$a = 2,\ b = 1,\ m = 13,\ x_0 = 2$$

• **ans:**

$$x_i = 2x_{i-1} \mod 13, \quad u_i = x_i/m$$

$x_0 = 1$ Then, the generated x_i are

$$x_i = 5, 11, 10, 8, 4, 9, 6, 0, 1, 3, 7, 2, 5, ...,$$

(a) Point 1 $(x, y) = (u_1, u_2) = (\frac{5}{13}, \frac{11}{13})$. Check the condition:

$$4(x - \frac{1}{2})^2 \le y \le (1 - x^3)?$$
$$0.0533 \le 0.8462 \le 0.9431?$$

Yes. The point is inside the region.

(b) Point 2 $(x, y) = (u_3, u_4) = (\frac{11}{13}, \frac{8}{13})$. Check the condition:

$$4(x - \frac{1}{2})^2 \le y \le (1 - x^3)?$$
$$0.2899 \le 0.6154 \le 0.5448?$$

No. The point is not inside the region.

(c) Point 3 $(x, y) = (u_5, u_6) = (\frac{4}{13}, \frac{9}{13})$. Check the condition:

$$4(x - \frac{1}{2})^2 \le y \le (1 - x^3)?$$
$$0.1479 \le 0.6923 \le 0.9709?$$

Yes. The point is inside the region.

(d) Point 4 $(x, y) = (u_7, u_8) = (\frac{6}{13}, \frac{0}{13})$. Check the condition:

$$4(x - \frac{1}{2})^2 \le y \le (1 - x^3)?$$
$$0.0059 \le 0 \le 0.9017?$$

No. The point is not inside the region.

(e) Point 5 $(x, y) = (u_9, u_{10}) = (\frac{1}{13}, \frac{3}{13})$. Check the condition:

$$4(x - \frac{1}{2})^2 \le y \le (1 - x^3)?$$
$$0.7160 \le 0.2308 \le 0.9995$$

No. The point is not inside the region.

Check the condition:

$$4(x - \frac{1}{2})^2 \le y \le (1 - x^3)?$$
$$0.0059 \le 0.1538 \le 0.8439?$$

Yes. The point is inside the region.

Out of $N = 6$ points, 3 of them are inside the region. So

$$A = \int\int 1 dA \simeq \frac{\text{\# pts inside}}{N} = \frac{3}{6} = 0.5$$

19. Find (1) the Fourier transform, (2) the fast Fourier transform

$$x = [0, 1, 0, -1]$$

- **<u>ans:</u>**

$$n = 4; \omega = e^{-i2\pi/n} = -i$$

$$F_n = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$y = F_n x = \begin{pmatrix} 0 \\ -i \\ 0 \\ i \end{pmatrix}$$

(skip the portion between these two lines.) Idea: $\omega^4 = 1$, (noting even rows repeating)

$$y = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} x$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega\omega^2 \\ 1 & \omega^2 & 1 & \omega^2(1) \\ 1 & \omega^3 & \omega^2 & \omega^3(\omega^2) \end{pmatrix} x$$

$$\mu = \omega^2, \ M_2 = \begin{pmatrix} \mu^0 & \mu^0 \\ \mu^0 & \mu^1 \end{pmatrix}$$

$$u = M_2 \begin{pmatrix} x_0 \\ x_2 \end{pmatrix}, \ v = M_2 \begin{pmatrix} x_1 \\ x_3 \end{pmatrix},$$

$$y = \frac{1}{\sqrt{4}}\begin{pmatrix} u_0 + \omega^0 v_0 \\ u_1 + \omega^1 v_1 \\ u_0 + \omega^2 v_0 \\ u_1 + \omega^3 v_1 \end{pmatrix}$$

Now,

$$\omega = -i, \ \mu = \omega^2 = -1$$

$$M_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$u = M_2 \begin{pmatrix} x_0 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ v = M_2 \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

$$y = \frac{1}{\sqrt{4}}\begin{pmatrix} u_0 + \omega^0 v_0 \\ u_1 + \omega^1 v_1 \\ u_0 + \omega^2 v_0 \\ u_1 + \omega^3 v_1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} u_0 + v_0 \\ u_1 - iv_1 \\ u_0 - 1v_0 \\ u_1 + iv_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -i \\ 0 \\ i \end{pmatrix}$$