M353 Study Guide for the Final (S. Zhang) .

1. Convert $a = 4.125$ and $b = 19/7$ to IEEE doubles. Then find the IEEE double form for $a + b$ (using chopping after bit 52.)

2. Find an interval of length one that contains a root.

$$x^4 = x^2 + 2$$

Then do two steps of bisection method. How many steps of bisection iterations are need to reach 10^{-10} accuracy?

3. For finding the root of the function:

$$f(x) = x^2 - 4x - 12$$

   (a) Do 3 steps of the Newton's method, $p_0 = 3$. Find the errors and use the data to show the method is a second-order one. Then find the fixed point, and the convergence order and rate there.
   (b) Do 3 steps of the secant method, $p_0 = 2$, $p_1 = 8$.
   (c) Do 3 steps of the false position method, given the initial interval $[2, 8]$.
   (d) Construct a function with p_0 and p_1 ($p_0 < p_1$) so that p_3 in the secant method is different from the second iterate in the false position method, where the initial interval is $[p_0, p_1]$.

4. Find convergence order and the rate of the Newton's method

$$32x^3 - 32x^2 - 6x + 9 = 0, \ r = -1/2, 3/4$$

5. Solve the following system $(A|b)$

$$\begin{pmatrix} 1 & -3 & 3 & | & 4 \\ -2 & 0 & 1 & | & -1 \\ 3 & -1 & -1 & | & 2 \end{pmatrix}$$

   (a) by GE without pivoting,
   (b) by finding $A = LU$ and using it
   (c) by GE with partial pivoting,
   (d) by finding $PA = LU$ and using it

6. Let

$$A = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

   (a) Find x_2 if Jacobi iteration is used.
   (b) Find the exact solution x and errors for the above Jacobi iteration, $\|x - x_i\|_\infty$, $i = 0, 1, 2$.
   (c) Find the error reduction bound for the Jacobin iteration, $\|R_j\|_\infty$. Check the error reduction data above.
   (d) Find x_2 if Gauss–Seidel iteration is used.
   (e) Find the error reduction bound for the Gauss–Seidel iteration, $\|R_{gs}\|_\infty$.

7. Find the $P_3(x)$ interpolation by
(1) solving equations for unknown coefficients,
(2) Lagrange nodal basis,
(3) Newton's divided differences.

x_i	-1	0	1	2
y_i	0	-2	0	0

8. Find the least-squares solution each problem.

$$\begin{pmatrix} -2 & -1 \\ 1 & 1 \\ 3 & -1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 0 \\ 14 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 & 3 \\ -1 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 21 \\ 9 \end{pmatrix}.$$

9. Let $f(x) = x + \sin x$, $x_0 = 0.5$.
   (a) Approximate $f'(x_0)$ by the (3-point) central differences with $h = 0.1$ and $h = 0.2$. Find the error bound for $h = 0.1$. Find the Richardson extrapolation. Check all three errors.
   (b) Using the (3-point) central differences with $h = 0.1$ and $h = 0.2$, Find the error bound for $h = 0.1$. Find the extrapolation to approximate $f''(x_0)$. Check all three errors.

10. Compute $\int_0^2 5x^4 \, dx$.
   (a) Use the trapezoidal rule with $m = 1(h = 2)$, and $m = 2$. Find the extrapolation. Find all three errors.
   (b) Use the mid-point rule with $m = 1(h = 2)$, and $m = 2$. Find the extrapolation. Find all three errors.
   (c) Use the Simpson's rule with $m = 1(h = 1)$, and $m = 2$. Find the extrapolation. Find all three errors.

11. Find Romberg R_{33} and the error for

$$\int_1^3 6x^5 \, dx.$$

12. Compute Gauss integration G_2, G_3 ($x_i = \pm\sqrt{.6}, 0$, $c_i = \frac{5}{9}, \frac{8}{9}$) for

$$\int_1^2 x^4 + 1 \, dx.$$

13. Find the exact solution and $y(1)$.
(1) Apply Euler method with $h = 1/2$ and $h = 1/4$ for $y(1)$ and the extrapolation, find the three errors.
(2) Apply backward Euler with $h = 1$ and $h = 1/4$ for $y(1)$ and the extrapolation, find the three errors.

$$y' = t - y, \ y(0) = 0$$

14. Solve the boundary value problem (find approximate values $x(1)$) :

$$x'' - x' = 4 - 4t, \qquad x(0) = 1, \quad x(2) = 9$$

(1) By linear Euler shooting with $h = 1$
(2) By nonlinear Euler shooting with $h = 1$, and 2 bisections starting with $x'(0) \in [0, 2]$ (4 shootings).
(3) By the finite difference method with $h = 1$.
The exact solution is $x = 2t^2 + 1$ (no use other than checking)

15. Solving the heat equation by the finite difference method. $u_t = 4u_{xx}$ for $t \in (0, 0.1)$ and $x \in (0, 3)$ with initial and boundary values

$$u(x,0) = 2(3-x)x, \quad u(0,t) = 0, \quad u(3,t) = 0.$$

Let $\Delta t = 0.1$. Find $u(x_i, 0.1)$ by the following methods with $h = 1$.

(a) the Euler (explicit) method.

(b) the backward Euler (implicit) method.

(c) the Crank-Nicholson method.

16. Solving the wave equation by the finite difference method.

$$u_{tt} = 4u_{xx}, \quad t \in (0, .4), \; x \in (0,1)$$

with initial and boundary values

$$u(x,0) = 4x(1-x), \quad u_t(x,0) = -x,$$

$$u(0,t) = 0, \quad u(1,t) = 0.$$

(a) Let $h = \Delta x = \frac{1}{2}$, $k = \Delta t = 0.1$. Find $u(x_i, 0.4)$.

(b) Let $h = \Delta x = \frac{1}{4}$, $k = \Delta t = 0.1$. Find $u(x_i, 0.2)$.

17. Solve the finite difference equations for the Laplace equation with $h = 1$.

$$u_{xx} + u_{yy} = 2 + 2x, \; 0 \le x, y \le 2$$

with the boundary condition

$$u_x(0,y) = y^2, \qquad u(2,y) = 2y^2 + 4$$
$$u(x,0) = x^2, \qquad u(x,2) = 4x + x^2.$$

18. Approximate the area bounded by

$$4(x - \frac{1}{2})^2 \le y \le (1 - x^3)$$

by the Monte Carlo method with N=6 and the linear congruential generator:

$$a = 2, \; b = 1, \; m = 13, \; x_0 = 2$$

19. Find (1) the Fourier transform, (2) the fast Fourier transform

$$x = [0, 1, 0, -1]$$