M353 Study Guide 3 (S. Zhang) .

1. Compute $\int_0^2 5x^4\,dx$.

60.281

(a) Use the trapezoidal rule with $m = 1(h = 2)$, and $m = 2$. Find the extrapolation. Find all three errors.

(b) Use the mid-point rule with $m = 1(h = 2)$, and $m = 2$. Find the extrapolation. Find all three errors.

(c) Use the Simpson's rule with $m = 1(h = 1)$, and $m = 2$. Find the extrapolation. Find all three errors.

- **ans:**

Exact value

$$\int_0^2 5x^4\,dx = 32$$

(a) Trapezoidal rule:
$m = 1$, $h = 2$:

$$\begin{aligned} T_h &= \frac{h}{2}(f_0 + f_1) \\ &= \frac{h}{2}(f(0) + f(2)) \\ &= 80 \\ err &= -48 \end{aligned}$$

$m = 2$, $h = 1$:

$$\begin{aligned} T_h &= \frac{h}{2}(f_0 + 2f_1 + f_2) \\ &= \frac{h}{2}(f(0) + 2f(1) + f(2)) \\ &= 45 \\ err &= -13 \end{aligned}$$

Extrapolation (second order)

$$\begin{aligned} extrap &= \frac{2^2 T_{m=2} - T_{m=1}}{2^2 - 1} \\ &= 33.333333 \\ err &= -1.3333333 \end{aligned}$$

(b) Midpoint rule:
$m = 1$, $h = 2$:

$$\begin{aligned} M_h &= h(f_{1/2}) = \frac{h}{2}(f(1)) \\ &= 10 \\ err &= 22 \end{aligned}$$

$m = 2$, $h = 1$:

$$\begin{aligned} M_h &= h(f_{1/2} + f_{3/2}) \\ &= h(f(0.5) + f(1.5)) \\ &= 25.625 \\ err &= 6.375 \end{aligned}$$

Extrapolation (second order)

$$\begin{aligned} extrap &= \frac{2^2 M_{m=2} - M_{m=1}}{2^2 - 1} \\ &= 30.8333333 \\ err &= 1.1666666 \end{aligned}$$

(c) Simpson's rule: $h = 1$:

$$\begin{aligned} S_h &= \frac{h}{3}(f(0) + 4f(1) + f(2)) \\ &= 33.33333333 \\ err &= -1.33333 \end{aligned}$$

$h = 1/2$:

$$\begin{aligned} S_h &= \frac{h}{3}(f(0) + 4f(\frac{1}{2}) + 2f(1) + 4f(\frac{3}{2}) + f(2)) \\ &= 32.083333333 \\ err &= -0.08333333 \end{aligned}$$

Extrapolation (4th order)

$$\begin{aligned} extrap &= \frac{2^4 S_{m=2} - S_{m=1}}{2^4 - 1} \\ &= 32 \\ err &= 0 \end{aligned}$$

2. Find Romberg R_{33} and the error for

62.61

$$\int_1^3 6x^5\,dx.$$

- **ans:**

$$\begin{aligned} R_{11} = T_{m=1} &= h[\frac{f_0}{2} + \frac{f_1}{2}] \\ &= 2[\frac{6(1^5)}{2} + \frac{6(3^5)}{2}] = 1464 \\ R_{21} = T_{m=2} &= \frac{R_{11}}{2} + hf_1 \\ &= \frac{R_{11}}{2} + 1(6(2^5)) = 934 \\ R_{31} = T_{m=4} &= \frac{R_{21}}{2} + h[f_1 + f_3] \\ &= 777.75 \\ R_{22} &= \frac{4R_{21} - R_{11}}{3} = 744 \\ R_{32} &= \frac{4R_{31} - R_{21}}{3} = 729 \\ R_{33} &= \frac{16R_{32} - R_{22}}{16 - 1} = 728 \end{aligned}$$

Exact

$$\int_1^3 6x^5\,dx = 3^6 - 1^6 = 728$$

Error is 0.

3. Determine the smallest integer j and N, for which the
62.6 Romberg integral R_{jj} (starting from R_{11}) and the Gauss-Legendre integral G_N is the exact, respectively.

$$\int_2^{12} 77x^{14} - x^3\,dx$$

• **ans:** (1) For Romberg integrals, $R(J,J)$ are exact for polynomial of degree $2j-1 \geq 14$, 8.
(2) For Gauss-Legendre integrals, G_N are exact for polynomial of degree $2N-1$. $2N-1 \geq 14$, $N=8$.

4. Compute the integral by adaptive quadrature with tolerance
63.14 0.2

$$\int_1^2 4x^3\,dx.$$

• **ans:**

$$a=1,\ b=2.\ \text{remain the same.}$$

(a) On $(1,2)$,

$$\begin{aligned} T_{h=1} &= 18, \\ T_{h=1/2} &= 15.75 \\ tol &= 3\cdot TOL\cdot\frac{b_1-a_1}{b-a} = 0.6 \\ T_{h=1}-T_{h=1/2} &= 2.25 \end{aligned}$$

Too big. Not accepted. We cut the interval into 2.

(b) On $(1,3/2)$,

$$\begin{aligned} T_{h=1/2} &= 4.375, \\ T_{h=1/4} &= 4.1406 \\ tol &= 3\cdot TOL\cdot\frac{b_1-a_1}{b-a} = 0.3 \\ T_{2h}-T_h &= 0.2344 \end{aligned}$$

Accepted, the difference is smaller than the control.

$$T_{(1,3/2)} = T_{h=1/4} = 4.1406$$

(c) On $(3/2,2)$,

$$\begin{aligned} T_{h=1/2} &= 11.375 \\ T_{h=1/4} &= 11.0469 \\ tol &= 3\cdot TOL\cdot\frac{b_1-a_1}{b-a} = 0.3 \\ T_{2h}-T_h &= 0.3281 \end{aligned}$$

Not accepted. Just slightly too big.

(d) On $(3/2,7/4)$,

$$\begin{aligned} T_{h=1/4} &= 4.3672 \\ T_{h=1/8} &= 4.3291 \\ tol &= 3\cdot TOL\cdot\frac{b_1-a_1}{b-a} = 0.15 \\ T_{2h}-T_h &= 0.0381 \end{aligned}$$

Accepted.

$$T_{(3/2,7/4)} = T_{h=1/8} = 4.3291$$

(e) On $(7/4,2)$,

$$\begin{aligned} T_{h=1/4} &= 6.6797, \\ T_{h=1/8} &= 6.6357 \\ tol &= 3\cdot TOL\cdot\frac{b_1-a_1}{b-a} = 0.15 \\ T_{2h}-T_h &= 0.0439 \end{aligned}$$

Accepted.

$$T_{(7/4,2)} = T_{h=1/8} = 6.6357$$

(f) Together

$$\begin{aligned} T &= T_{(1,3/2)} + T_{(3/2,7/4)} + T_{(7/4,2)} \\ &= 15.1055 \end{aligned}$$

Checking:

$$\int_1^2 4x^3\,dx = 15$$

Error:

$$|\int_1^2 4x^3\,dx - T| = |15-15.1055| = .1055 < .2.$$

Yes.

5. Compute Gauss integration G_2, G_3 ($x_i = \pm\sqrt{.6}, 0, c_i = \frac{5}{9}, \frac{8}{9}$
64.4) for

$$\int_1^2 x^4 + 1\,dx.$$

• **ans:** Formula (change variables – must apply Gauss formula on $[-1,1]$)

$$\int_a^b f(z)dz = \int_{-1}^1 f(\frac{a+b}{2} + u\frac{b-a}{2})\frac{b-a}{2}du$$

$$\int_{-1}^1 \left(\left(\frac{u+3}{2}\right)^4 + 1\right)\frac{du}{2}$$

$$\begin{aligned} G_2 &= \frac{\left(\frac{-\frac{1}{\sqrt{3}}+3}{2}\right)^4 + 1}{2} + \frac{\left(\frac{\frac{1}{\sqrt{3}}+3}{2}\right)^4 + 1}{2} \\ &= 7.19444418 \end{aligned}$$

$$\begin{aligned} G_3 &= \frac{5}{9}\frac{\left(\frac{-\sqrt{.6}+3}{2}\right)^4 + 1}{2} + \frac{8}{9}\frac{\left(\frac{3}{2}\right)^4 + 1}{2} \\ &+ \frac{5}{9}\frac{\left(\frac{\sqrt{.6}+3}{2}\right)^4 + 1}{2} \\ &= 7.19999988 \end{aligned}$$

The exact integral is 7.2.

6. Solve the IVP. Find $y(0.5)$,

66.401

$$y' + 2y = 2e^{-t}; \quad y(0) = 2.$$

Apply Euler method with $h = 1/2$ and $h = 1/4$. Extrapolate the solutions for $y(1/2)$. Find three errors.

- **ans:** 1st order linear:

$$\mu = e^{\int p} = e^{2t}$$

sol:

$$y = \mu^{-1} \int \mu q$$

$$y = e^{-2t} \int 2e^{-t} e^{2t} dt$$

$$y = e^{-2t}(2e^t + C)$$

$$y = e^{-t},\ y(1/2) = 1.2131$$

$$y_{i+1} = y_i + hf_i = y_i + hf(t_i, y_i)$$

$$y' = f(t, y) = -2y + 2e^{-t}$$

Euler for $h = 1/2$,

$$t_0 = 0,\ y_0 = 2,$$

$$\begin{aligned} y_1 &= 2 + \frac{1}{2}(-2(2) + 2e^0) \\ &= 1,\ err = 0.2131 \end{aligned}$$

Euler for $h = 1/4$,

$$t_0 = 0,\ y_0 = 2,$$

$$\begin{aligned} y_1 &= 2 + \frac{1}{4}(-2(2) + 2e^0) \\ &= 1.5, \end{aligned}$$

$$t_1 = 1/4,$$

$$y_2 = 1.1394,\ err = 0.0737$$

The error is roughly 1/2 of the previous one.

Extrapolation, the Euler method is a first order method (one step error $O(h^2)$ is of second order, though)

$$\begin{aligned} extrap &= \frac{2y_{h=1/4} - y_{h=1/2}}{2 - 1} \\ &= 1.2788,\ err = -0.0657 \end{aligned}$$

The extrapolated solution is slightly better. It should be much smaller if we did more steps.

7. Given an initial value problem

67.401

$$\begin{aligned} x'' - 4x = 8t, &\quad 0 < t < 0.2 \\ x(0) = 0, &\quad x'(0) = -2. \end{aligned}$$

(a) Find the exact solution.

(b) Convert the equation to a system. Use Euler's method with step size $h = 0.1$ and find the error.

(c) Use R-K method with step size $h = 0.2$ and find the error.

- **ans:** We find solution in two steps.

First, find x_H (for the homogeneous equation)

$$\begin{aligned} x'' - 4x &= 0 \\ r^2 - 4 &= 0 \\ (r + 2)(r - 2) &= 0,\ r = -2, 2 \\ x_H &= c_1 e^{-2t} + c_2 e^{2t} \end{aligned}$$

Step 2, find a particular solution for the nonhomogeneous equation:

$$\begin{aligned} x_p &= At + B \\ (0) - 4(Ax + B) &= 8t \\ A = -2,&\ B = 0 \end{aligned}$$

$$x = x_H + x_p = c_1 e^{-2t} + c_2 e^{2t} - 2t$$

By the initial condition, we get

$$x = -2t.$$

Next, we convert one equation to a system of first order equations. For a second order equation, we always let

$$y = x'$$

Then we put them together as a vector,

$$\begin{aligned} Z &= \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix} \\ Z' &= \begin{pmatrix} x' \\ x'' \end{pmatrix} \\ Z' &= F(t, Z) = \begin{pmatrix} y \\ 4x + 8t \end{pmatrix} \end{aligned}$$

Use Euler's method.

$$\begin{aligned} t_0 = 0; Z_0 &= \begin{pmatrix} 0 \\ -2 \end{pmatrix}; h = 0.1; \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= Z_0 + h \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.2 \\ -2 \end{pmatrix} \end{aligned}$$

We need two steps.

$$t_1 = t_0 + h = 0.1$$

$$
\begin{aligned}
Z_2 &= Z_1 + hF(t_1, Z_1)\\
&= Z_1 + h\begin{pmatrix} -2 \\ 4(-0.2)+8(0.1)\end{pmatrix}\\
&= Z_1 + h\begin{pmatrix} -2 \\ 0\end{pmatrix} = \begin{pmatrix} -0.4 \\ -2\end{pmatrix}\\
err &= x(0.2) - (Z_2)_1 = -0.4 - (-0.4) = 0
\end{aligned}
$$

Use the Runge-Kutta method.

$$
t_0 = 0; Z_0 = \begin{pmatrix} 0 \\ -2\end{pmatrix}; h = 0.2
$$

$$
\begin{aligned}
k_1 &= F(t_0, y_0)\\
&= F(t_0, \begin{pmatrix} 0 \\ -2\end{pmatrix}) = \begin{pmatrix} -2 \\ 0\end{pmatrix}\\
k_2 &= F(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1)\\
&= F(t_0 + \frac{h}{2}, \begin{pmatrix} -0.2 \\ -2\end{pmatrix}) = \begin{pmatrix} -2 \\ 0\end{pmatrix}\\
k_3 &= F(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2)\\
&= F(t_0 + \frac{h}{2}, \begin{pmatrix} -0.2 \\ -2\end{pmatrix}) = \begin{pmatrix} -2 \\ 0\end{pmatrix}\\
k_4 &= F(t_0 + h, y_0 + hk_3)\\
&= F(t_0 + h, \begin{pmatrix} -0.4 \\ -2\end{pmatrix}) = \begin{pmatrix} -2 \\ 0\end{pmatrix}\\
Z_1 &= y_0 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\\
&= y_0 + h\begin{pmatrix} -2 \\ 0\end{pmatrix} = \begin{pmatrix} -0.4 \\ -2\end{pmatrix}
\end{aligned}
$$

Error:

$$
x(0.2) - (Z_1)_1 = -0.4 - (-0.4) = 0
$$

8. Find the exact solution. Apply backward Euler with $h =$
71.11 $1/4$ for $y(1)$, find the error.

$$
y' = t - y,\ y(0) = 0
$$

- **ans:**

1st order linear: $y' + py = q$.

$$
\mu = e^{\int p} = e^t
$$

sol:

$$
\begin{aligned}
y &= \mu^{-1}\int \mu q = e^{-t}\int e^t t dt\\
&= e^{-t}(e^t t - e^t + C) = t - 1 + Ce^{-t}
\end{aligned}
$$

$$
y(0) = 0 \Rightarrow y = t - 1 + e^{-t}
$$

$$
y(1) = e^{-1} = 0.3678794411714
$$

Backward Euler: $y' = f(t, y)$.

$$
\begin{aligned}
y_1 &= y_0 + hf_1 = y_0 + hf(t_1, y_1)\\
&= y_0 + ht_1 - hy_1
\end{aligned}
$$

$$
y_1 = \frac{y_0 + ht_1}{1 + h}
$$

$$
y_{i+1} = \frac{y_i + ht_{i+1}}{1 + h}
$$

$$
h = 1/4, ;\ y_0 = 0,\ t_0 = 0
$$

$$
\begin{aligned}
t_1 = 0.25;\quad y_1 &= \frac{0 + 0.25(0.25)}{1.25}\\
&= 0.05\\
err = y(t_i) - y_i &= -0.0212
\end{aligned}
$$

$$
\begin{aligned}
i = 2;\ t_i &= 0.5\\
y_i &= 0.14\\
err = y(t_i) - y_i &= -0.0335
\end{aligned}
$$

$$
\begin{aligned}
i = 3;\ t_i &= 0.75\\
y_i &= 0.262\\
err = y(t_i) - y_i &= -0.0396
\end{aligned}
$$

$$
\begin{aligned}
i = 4;\ t_i &= 1\\
y_i &= 0.4096\\
err = y(t_i) - y_i &= -0.0417
\end{aligned}
$$

9. Solve the boundary value problem (BVP) by shooting
72.3001 method:

$$
\begin{aligned}
&x'' - 3x' + 2x = 4t^2 - 12t + 6,\\
&x(0) = 1, \quad x(1) = 3
\end{aligned}
$$

(a) Find the exact solution. $(x_H + x_P)$

(b) Find the exact solution $u = x(t)$ for IVP (shooting 1)

$$
x'' - 3x' + 2x = 4t^2 - 12t + 6, \quad x(0) = 1, \quad x'(0) = 0
$$

(c) Find the exact solution $v = x(t)$ for IVP (shooting 2)

$$
x'' - 3x' + 2x = 0 \qquad x(0) = 0, \qquad x'(0) = 1
$$

(d) Then combine u and v by the shooting method to get the exact solution of the original BVP.

(e) (Linear Euler Shooting) Convert the above two shooting problems to systems of two equations, and apply the Euler method with $h = 1/2$ to them. Combine the two discrete solutions and find the approximate value of $x(1/2)$. Find the error.

(f) (Linear Euler Shooting) Solve the BVP by the Euler shooting method with $h = 1/4$. Find the error at $t = 1/2$. Extrapolate two linear shooting results to get a better approximation of $x(1/2)$.

(g) (Linear shooting) Solve the BVP by the Runge-Kutta shooting method with $h = 1/2$. Find the error at $t = 1/2$.

(h) (**Nonlinear** Euler shooting) Apply the 2 steps (5 shootings) of the bisection method with Euler discretization of grid size $h = 1/2$, using starting interval $x'(0) \in [-1, 1]$. Find approximate $x(1/2)$ and its error.

(i) Solve the finite difference equations for the boundary value problem with grid size $h = 1/2$ to get an approximation of $x(1/2)$. Find the error.

(j) Solve the BVP by the finite difference method with $h = 1/4$. Find the error at $t = 1/2$.

(k) Use extrapolation on the last two $x(1/2)$ values to find a new $x(1/2)$ and the error.

- **ans:**

(a) Find x_H and x_P:
$r^2 - 3r + 2 = 0$, $r = 1, 2$, $x_H = C_1e^t + C_2e^{2t}$,

$$x_P = At^2 + Bt + C \Rightarrow x_P = 2t^2 + 1$$

$$x = x_H + x_P = C_1e^t + C_2e^{2t} + 2t^2 + 1.$$

By boundary conditions,

$$x = 2t^2 + 1.$$

$$x(\frac{1}{2}) = 1.5.$$

(b) Exact shooting 1.
Find x_H (same): $r^2 - 3r + 2 = 0$, $r = 1, 2$, $x_H = C_1e^t + C_2e^{2t}$,
Find x_P:

$$x_P = At^2 + Bt + C \Rightarrow x_P = 2t^2 + 1$$

$$u = x_H + x_P = C_1e^t + C_2e^{2t} + 2t^2 + 1.$$

By initial conditions,

$$\begin{aligned} 1 &= C_1 + C_2 + 1 \\ 0 &= C_1 + 2C_2 \end{aligned}$$

$$u = 2t^2 + 1.$$

(c) Exact shooting 2.
Find x_H (same): $r^2 - 3r + 2 = 0$, $r = 1, 2$, $x_H = C_1e^t + C_2e^{2t}$,

$$v = x = x_H = C_1e^t + C_2e^{2t}.$$

By initial conditions,

$$\begin{aligned} 0 &= C_1 + C_2 \\ 1 &= C_1 + 2C_2 \end{aligned}$$

$$v = -e^t + e^{2t}.$$

(d) Combining (2 exact linear shootings) u and v,

$$\begin{aligned} x &= u + \frac{x(1) - u(1)}{v(1)} v \\ &= u\frac{3 - 3}{-e + e^2} v = u = 2t^2 + 1. \end{aligned}$$

(e) Linear Euler Shooting with $h = 1/2$.
Linear Euler Shooting 1 with $h = 1/2$. <u>For u:</u>

$$\begin{aligned} x' &= y \\ y' &= 3y - 2x + 4t^2 - 12t + 6 \end{aligned}$$

$$h = 1/2,\ U = Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}\ U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$U' = F(t, U) = \begin{pmatrix} y \\ 3y - 2x + 4t^2 - 12t + 6 \end{pmatrix}$$

$$\begin{aligned} U_1 &= U_0 + hF(t_0, U_0) \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 3(0) - 2(1) + 4(0) - 12(0) + 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} U_2 &= U_1 + hF(t_1, U_1) \\ &= \begin{pmatrix} 2 \\ 4.5 \end{pmatrix} \end{aligned}$$

$$u(1) \sim (U_2)_1 = 2$$

Linear Euler Shooting 2 with $h = 1/2$. <u>For v:</u>

$$\begin{aligned} x' &= y \\ y' &= 3y - 2x \end{aligned}$$

$h = 1/2$,

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix},\quad V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V' = F(t, V) = \begin{pmatrix} y \\ 3y - 2x \end{pmatrix}$$

$$\begin{aligned} V_1 &= V_0 + hF(t_0, V_0) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 3(1) - 2(0) \end{pmatrix} = \begin{pmatrix} .5 \\ 2.5 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
V_2 &= V_1 + hF(t_1, V_1) \\
&= \begin{pmatrix} 1.75 \\ 5.75 \end{pmatrix}
\end{aligned}
$$

$$v(1) \sim (V_2)_1 = 1.75$$

Combine Linear Euler Shootings with $h = 1/2$:

$$
\begin{aligned}
x(t) &= u + \frac{\beta - u(b)}{v(b)} v \\
&= u + \frac{3-2}{1.75} v \\
&= u + 0.57142v
\end{aligned}
$$

In particular,

$$
\begin{aligned}
x(1/2) &\sim u(1/2) + 0.57142v(1/2) = (U_1)_1 + 0.57142(V_1)_1 \\
&= 1.2857
\end{aligned}
$$

Error is 0.2142.

(f) Linear Euler Shooting with $h = 1/4$.
Linear Euler Shooting 1 with $h = 1/4$. For u:

$$
\begin{aligned}
x' &= y \\
y' &= 3y - 2x + 4t^2 - 12t + 6
\end{aligned}
$$

$h = 1/4$, $U = Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$ $U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$U' = F(t, U) = \begin{pmatrix} y \\ 3y - 2x + 4t^2 - 12t + 6 \end{pmatrix}$$

$$
\begin{aligned}
U_1 &= U_0 + hF(t_0, U_0) \\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}
$$

$$U_2 = U_1 + hF(t_1, U_1) = \begin{pmatrix} 1.25 \\ 2.0625 \end{pmatrix}$$

$$U_3 = U_2 + hF(t_2, U_2) = \begin{pmatrix} 1.765625 \\ 3.234375 \end{pmatrix}$$

$$U_4 = U_4 + hF(t_3, U_3) = \begin{pmatrix} 2.57421875 \\ 4.58984375 \end{pmatrix}$$

$$u(1) \sim (U_4)_1 = 2.57421875$$

Linear Euler Shooting 2 with $h = 1/4$. For v:

$$
\begin{aligned}
x' &= y \\
y' &= 3y - 2x
\end{aligned}
$$

$h = 1/4$,

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V' = F(t, V) = \begin{pmatrix} y \\ 3y - 2x \end{pmatrix}$$

$$
\begin{aligned}
V_1 &= V_0 + hF(t_0, V_0) \\
&= \begin{pmatrix} .25 \\ 1.75 \end{pmatrix}
\end{aligned}
$$

$$V_2 = V_1 + hF(t_1, V_1) = \begin{pmatrix} 0.6875 \\ 2.9375 \end{pmatrix}$$

$$V_3 = V_2 + hF(t_2, V_2) = \begin{pmatrix} 1.421875 \\ 4.796875 \end{pmatrix}$$

$$V_4 = V_3 + hF(t_3, V_3) = \begin{pmatrix} 2.62109375 \\ 7.68359375 \end{pmatrix}$$

$$v(1) \sim (V_4)_1 = 2.62109375$$

Combine 2 Linear Euler Shootings with $h = 1/4$:

$$
\begin{aligned}
x(t) &= u + \frac{\beta - u(b)}{v(b)} v \\
&= u + \frac{3 - 2.57421875}{2.62109375} v \\
&= u + .162444v
\end{aligned}
$$

In particular,

$$
\begin{aligned}
x(1/2) &\sim u(1/2) + .162444v(1/2) \\
&= (U_2)_1 + .162444(V_2)_1 \\
&= 1.25 + .162444(0.6875) = 1.36168
\end{aligned}
$$

Error is 0.1383. (smaller than that of $h = 1/2$.)

(g) Linear Runge-Kutta Shooting with $h = 1/2$.
Linear Runge-Kutta Shooting 1 with $h = 1/2$. For u:

$$
\begin{aligned}
x' &= y \\
y' &= 3y - 2x + 4t^2 - 12t + 6
\end{aligned}
$$

$h = 1/2$, $U = Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$ $U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$U' = F(t, U) = \begin{pmatrix} y \\ 3y - 2x + 4t^2 - 12t + 6 \end{pmatrix}$$

$$k_1 = F(t_0, U_0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$
$$k_2 = F(t_0 + \frac{h}{2}, U_0 + \frac{h}{2}k_1) = \begin{pmatrix} 1 \\ 4.25 \end{pmatrix}$$
$$k_3 = F(t_0 + \frac{h}{2}, U_0 + \frac{h}{2}k_2) = \begin{pmatrix} 1.0625 \\ 3.9375 \end{pmatrix}$$
$$k_4 = F(t_0 + h, U_0 + hk_3) = \begin{pmatrix} 1.96875 \\ 3.84375 \end{pmatrix};$$
$$U_1 = U_0 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \begin{pmatrix} 1.5078125 \\ 2.018229167 \end{pmatrix};$$
$$t_1 = 0.5$$
$$k_1 = F(t_1, U_1) = \begin{pmatrix} 2.018229 \\ 4.0390625 \end{pmatrix}$$
$$k_2 = F(t_1 + \frac{h}{2}, U_1 + \frac{h}{2}k_1) = \begin{pmatrix} 3.02799479 \\ 4.30924479 \end{pmatrix}$$
$$k_3 = F(t_1 + \frac{h}{2}, U_1 + \frac{h}{2}k_2) = \begin{pmatrix} 3.0955403 \\ 4.0069986 \end{pmatrix}$$
$$k_4 = F(t_1 + h, U_1 + hk_3) = \begin{pmatrix} 4.0217285 \\ 3.954020 \end{pmatrix};$$
$$U_2 = U_1 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \begin{pmatrix} 3.031731 \\ 4.07035997 \end{pmatrix};$$

$$u(1) \sim (U_2)_1 = 3.031731$$

Linear Runge-Kutta Shooting 2 with $h = 1/2$. For v:

$$x' = y$$
$$y' = 3y - 2x$$

$h = 1/2$,

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V' = F(t, V) = \begin{pmatrix} y \\ 3y - 2x \end{pmatrix}$$

$$V' = F(t, V) = \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}$$

$$k_1 = F(t_0, V_0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$k_2 = F(t_0 + \frac{h}{2}, V_0 + \frac{h}{2}k_1) = \begin{pmatrix} 1.75 \\ 4.75 \end{pmatrix}$$
$$k_3 = F(t_0 + \frac{h}{2}, V_0 + \frac{h}{2}k_2) = \begin{pmatrix} 2.1875 \\ 5.6875 \end{pmatrix}$$
$$k_4 = F(t_0 + h, V_0 + hk_3) = \begin{pmatrix} 3.84375 \\ 9.34375 \end{pmatrix};$$
$$V_1 = V_0 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \begin{pmatrix} 1.05989583 \\ 3.7682291 \end{pmatrix};$$
$$t_1 = 0.5$$
$$k_1 = F(t_1, V_1) = \begin{pmatrix} 3.7682291 \\ 9.18489583 \end{pmatrix}$$
$$k_2 = F(t_1 + \frac{h}{2}, V_1 + \frac{h}{2}k_1) = \begin{pmatrix} 6.064453125 \\ 14.189453125 \end{pmatrix}$$
$$k_3 = F(t_1 + \frac{h}{2}, V_1 + \frac{h}{2}k_2) = \begin{pmatrix} 7.315592 \\ 16.794759 \end{pmatrix}$$
$$k_4 = F(t_1 + h, V_1 + hk_3) = \begin{pmatrix} 12.16560872 \\ 27.06144205 \end{pmatrix};$$
$$V_2 = V_1 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \begin{pmatrix} 4.6177232 \\ 11.9527926 \end{pmatrix};$$

$$v(1) \sim (V_2)_1 = 4.6177232$$

Combine Linear Runge-Kutta Shootings with $h = 1/2$:

$$\begin{aligned} x(t) &= u + \frac{\beta - u(b)}{v(b)}v \\ &= u + \frac{3 - 3.031731}{4.6177232}v \\ &= u - 0.0068716763v \end{aligned}$$

In particular,

$$\begin{aligned} x(1/2) &\sim u(1/2) - 0.0068716763v(1/2) \\ &= (U_1)_1 - 0.0068716763(V_1)_1 \\ &= 1.5078125 - 0.0068716763 * 1.05989583333333 \\ &= 1.500529238 \end{aligned}$$

Error is -0.000529238. (much smaller)

(h) Nonlinear Euler shooting with $h = 1/2$.

Here we repeatedly doing the first shooting in the linear shooting method – and use bisection to adjust the shooting angle $x'(0)$ for better landing on the target.

Convert the second order equation to a first order system:

$$x' = y$$
$$y' = x'' = 3y - 2x + 4t^2 - 12t + 6$$

$$Z = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$Z' = F(t, Z) = \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} y \\ 3y - 2x + 4t^2 - 12t + 6 \end{pmatrix}$$

Discretization:

$$h = 1,\ t_0 = 0,\ t_1 = 1,\ t_2 = 2$$

The key job is to find $x'(0)$. So we start with two tries $x'(0) = -1$ and $x'(0) = 1$.

(Note that, for linear shooting problems, two tries are enough to find the correct shooting. But for nonlinear shooting problems, we have to try many many times, each time, we try again shooting in the middle — bisection method.)

Shooting 1: $x' = -1$ (the lower end point)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \\ t_1 &= t_0 + h = 1/2 \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ Z_1 &+ \frac{1}{2}\begin{pmatrix} -1/2 \\ -3/2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ * \end{pmatrix} \end{aligned}$$

We shoot at $x = 1/4$, too low (target 3).

Shooting 2: $x' = 1$ (the high end point)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} 3/2 \\ 9/2 \end{pmatrix} \\ t_1 &= t_0 + h = 1/2 \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ Z_1 &+ \frac{1}{2}\begin{pmatrix} 9/2 \\ 23/2 \end{pmatrix} = \begin{pmatrix} 15/4 \\ * \end{pmatrix} \end{aligned}$$

We shoot at $x = 15/4$, too high (target 3).

So, we know we need to choose a shooting angle $x'(0)$ in between. We apply bisection method.

Shooting 3: $x' = \frac{-1+1}{2} = 0$ (the middle point)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ t_1 &= t_0 + h = 1/2 \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ Z_1 &+ \frac{1}{2}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ * \end{pmatrix} \end{aligned}$$

We shoot at $x = 2$, too low (target 3).
What do we try next? Between $(-1, 0)$ or between $(0, 1)$?
Since the target is between the shootings of angles 0 and 1, we use $(0, 1)$ for the next shooting.

Shooting 4: $x' = \frac{0+1}{2} = 0.5$ (the middle point)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1/2 \\ 11/2 \end{pmatrix} \\ &= \begin{pmatrix} 5/4 \\ 13/4 \end{pmatrix} \\ t_1 &= t_0 + h = 1/2 \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ Z_1 &+ \frac{1}{2}\begin{pmatrix} 13/4 \\ * \end{pmatrix} = \begin{pmatrix} 23/8 \\ * \end{pmatrix} \end{aligned}$$

We shoot at $x = 23/8$, a little too low (target 3).
What do we try next?
We finished two bisection iterations. We know the right shooting $x'(0)$ would be between 0.5 and 1. To find the $x(1/2)$ for the best shooting, we compute one more time.

Shooting 5: $x' = \frac{0.5+1}{2} = 0.75$ (the middle point)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 1 \\ 3/4 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 1 \\ 3/4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3/4 \\ 25/4 \end{pmatrix} \\ &= \begin{pmatrix} 11/8 \\ 31/8 \end{pmatrix} \\ t_1 &= t_0 + h = 1/2 \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ Z_1 &+ \frac{1}{2}\begin{pmatrix} 31/8 \\ * \end{pmatrix} = \begin{pmatrix} 53/16 \\ * \end{pmatrix} \end{aligned}$$

So we find only one approximate value for the x in the middle

$$x(1/2) \sim Z_{1,1} = 11/8 = 1.375$$

error:

$$x(\frac{1}{2}) - Z_{1,1} = 1.5 - 1.375 = 0.125.$$

(i) Finite difference method with $h = 1/2$.
Central difference:

$$x_1'' = \frac{x_0 - 2x_1 + x_2}{h^2}$$

$$x_1' = \frac{x_2 - x_0}{2h}$$

Only 1 unknown x_1. $x_0 = 1$, and $x_2 = 3$. At $t = t_1 = 1/2$,

$$\frac{x_0 - 2x_1 + x_2}{h^2} - 3\frac{x_2 - x_0}{2h} + 2x_1 = 4\frac{1}{2^2} - 12\frac{1}{2} + 6$$

$$\begin{aligned} &- 2x_1 + 2h^2 x_1 \\ =&h^2(4\frac{1}{2^2} - 12\frac{1}{2} + 6) - x_0 - x_2 - 3\frac{x_0 h}{2} + 3\frac{x_2 h}{2} \\ -1.5x_1 &= -2.25 \\ \Rightarrow & x(1/2) \sim x_1 = 1.5 = x(1/2)! \end{aligned}$$

The error is 0! (Because the method is a second order one.)

(j) Finite difference method with $h = 1/4$.
3 unknowns x_1, x_2, x_3. Given $x_0 = 1$, and $x_4 = 3$. At $t = t_1 = 1/4$,

$$\frac{x_0 - 2x_1 + x_2}{h^2} - 3\frac{x_2 - x_0}{2h} + 2x_1 = 4\frac{1}{4^2} - 12\frac{1}{4} + 6$$

$$\begin{aligned} &- 2x_1 + x_2 - 3\frac{x_2 h}{2} + 2h^2 x_1 \\ =&h^2(4\frac{1}{4^2} - 12\frac{1}{4} + 6) - x_0 - 3\frac{x_0 h}{2} \end{aligned}$$

Similarly,

$$\begin{aligned} &x_1 - 2x_2 + x_3 - 3\frac{x_3 h}{2} + 3\frac{x_1 h}{2} + 2h^2 x_2 \\ =&h^2(4\frac{1}{2^2} - 12\frac{1}{2} + 6) \\ &x_2 - 2x_3 + x_3 + 3\frac{x_2 h}{2} + 2h^2 x_2 \\ =&h^2(4\frac{1}{4^2} - 12\frac{1}{4} + 6) - x_4 + 3\frac{x_4 h}{2} \end{aligned}$$

Write three equations in a system $Ax = b$:

$$\begin{pmatrix} -1.875 & 0.625 & \\ 1.375 & -1.875 & 0.625 \\ & 1.375 & -1.875 \end{pmatrix} x = \begin{pmatrix} -1.171875 \\ 0.0625 \\ -1.921875 \end{pmatrix}$$

$$x = A^{-1}b = \begin{pmatrix} 1.125 \\ 1.5 \\ 2.125 \end{pmatrix}$$

$$x(1/2) \sim x_2 = 1.5 = x(1/2)!$$

The error is 0 again.

(k) Extrapolation.
The method is a 2nd order method.

$$\begin{aligned} extrap &= \frac{2^2 x_{h=1/4}(\frac{1}{2}) - x_{h=1/2}(\frac{1}{2})}{2^2 - 1} \\ &= 1.5 = x(1/2) \end{aligned}$$

Exact again.

10. Solve the boundary value problem by the Euler shooting
72.15 method with $h = 1/2$ (find approximate values $x(1/2)$) :

$$x'' - x' = 2 - 2t, \qquad x(0) = 2, \quad x(1) = 3$$

Then solve it by the finite difference method with $h = 1/2$, and $h = 1/3$. The exact solution is given:

$$x = t^2 + 2$$

- **ans:** Exact solution is

$$x(1/2) = 2.25$$

Shooting 1 – For u:

$$x'' - x' = 2 - 2t, \; y' = x'' = y + 2 - 2t$$

$h = 1$,

$$U = Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$U_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

$$U' = F(t, U) = \begin{pmatrix} y \\ y + 2 - 2t \end{pmatrix}$$

$$\begin{aligned} U_1 &= U_0 + hF(t_0, U_0) \\ &= U_0 + h\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} U_2 &= U_1 + hF(t_1, U_1) \\ &= U_1 + h\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2 \end{pmatrix} \end{aligned}$$

Shooting 2 – For v:

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x'' - x' = 0,\ y' = x'' = y$$

$$V' = F(t, V) = \begin{pmatrix} y \\ y \end{pmatrix}$$

$$\begin{aligned} V_1 &= V_0 + hF(t_0, V_0) \\ &= V_0 + h \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} .5 \\ 1.5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} V_2 &= V_1 + hF(t_1, V_1) \\ &= V_1 + h \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 2.25 \end{pmatrix} \end{aligned}$$

Combine them:

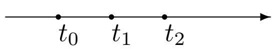
$$\begin{aligned} x(t) &= u + \frac{\beta - u(b)}{v(b)} v \\ &= u + \frac{3 - 2.5}{1.25} v \\ &= u + 0.4v \end{aligned}$$

In particular,

$$\begin{aligned} x(1/2) &\sim u(1/2) + 0.4v(1/2) = (U_1)_1 + 0.4(V_1)_1 \\ &= 2.2 \end{aligned}$$

Error is 0.05.

Finite difference for $h = 1/2$:

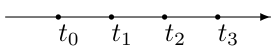


When $h = 1/2$, only 1 unknown x_1. x_0 and x_2 and known.

$$\begin{aligned} &\frac{x_0 - 2x_1 + x_2}{h^2} - \frac{x_2 - x_0}{2h} = 2 - 2t_1 \\ &(-2)x_1 = -4.5 \\ &x_1 = 2.25 \end{aligned}$$

The error is 0!

Finite difference for $h = 1/3$:



When $h = 1/3$, 2 unknowns x_1, x_2. x_0 and x_3 are the two given boundary values. The 2 equations are similar to the one above except h is different and all indices are increased by one.

$$\begin{aligned} \frac{x_0 - 2x_1 + x_2}{h^2} - \frac{x_2 - x_0}{2h} &= 2 - 2t_1 \\ \frac{x_1 - 2x_2 + x_3}{h^2} - \frac{x_3 - x_1}{2h} &= 2 - 2t_2 \end{aligned}$$

We would get the following linear system:

$$\begin{pmatrix} -2 & 0.8333 \\ 1.1667 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2.1852 \\ -2.4259 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2.111 \\ 2.4444 \end{pmatrix}$$

The errors are

$$\begin{aligned} x(1/3) - 2.111 &= (\frac{1}{3})^2 + 2 - 2.111 = 0 \\ x(2/3) - 2.444 &= (\frac{2}{3})^2 + 2 - 2.444 = 0 \end{aligned}$$