

**M353 Study Guide 2** (S. Zhang).

1. Solve the following system  $Ax = b$

- (a) by GE without pivoting, (no row switching, no row multiplication)
- (b) by finding  $A = LU$  and using it
- (c) by GE with partial pivoting, (row pivoting every step)
- (d) by finding  $PA = LU$  and using it

$$(A|b) = \left( \begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ -2 & 0 & 2 & 2 \\ 3 & -1 & -1 & 0 \end{array} \right)$$

• **ans:**

- (a)  $2r_1 + r_2, -3r_1 + r_3$ :

$$\left( \begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & -6 & 6 & 12 \\ 0 & 8 & -7 & -15 \end{array} \right)$$

$(8/6)r_2 + r_3$ :

$$\left( \begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & -6 & 6 & 12 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

- (b) By above  $2r_1 + r_2, -3r_1 + r_3$  and  $(8/6)r_2 + r_3$ :

$$L = \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 3 & -8/6 & 1 & \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & -3 & 2 \\ 0 & -6 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = LU$$

Solving the system in two steps:

$$Ly = b \quad y = \begin{pmatrix} 5 \\ 12 \\ 1 \end{pmatrix} \text{ (copy above)}$$

$$Ux = y \quad x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

- (c) Row switching: 1 3, (pivots: 1, -2, 3)

$$\left( \begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ -2 & 0 & 2 & 2 \\ 1 & -3 & 2 & 5 \end{array} \right) \begin{matrix} r_3 \\ r_2 \\ r_1 \end{matrix}$$

$(2/3)r_1 + r_2, (-1/3)r_1 + r_3$ :

$$\left( \begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ -2/3 & -0.66667 & 1.33333 & 2 \\ 1/3 & -2.66667 & 2.33333 & 5 \end{array} \right) \begin{matrix} r_3 \\ r_2 \\ r_1 \end{matrix}$$

Row switching: 2 3, (pivots: -0.66667, -2.66667)

$$\left( \begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1/3 & -2.66667 & 2.33333 & 5 \\ -2/3 & -0.66667 & 1.33333 & 2 \end{array} \right) \begin{matrix} r_3 \\ r_1 \\ r_2 \end{matrix}$$

$(-1/4)r_2 + r_3$ :

$$\left( \begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1/3 & -2.66667 & 2.33333 & 5 \\ -2/3 & 1/4 & 0.75 & 0.75 \end{array} \right) \begin{matrix} r_3 \\ r_1 \\ r_2 \end{matrix}$$

Backward substitution:

$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

- (d) From the row index column above, we get

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

From the boxed numbers, we get

$$L = \begin{pmatrix} 1 & & \\ 1/3 & 1 & \\ -2/3 & 1/4 & 1 \end{pmatrix}$$

And the left over matrix above gives us

$$U = \begin{pmatrix} 3 & -1 & -1 \\ -8/3 & 7/3 & \\ & & 3/4 \end{pmatrix}$$

Three steps in solving  $Ax = b$ .

$$Pz = b \quad z = Pb = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

$$Ly = b \quad y = \begin{pmatrix} 0 \\ 5 \\ 3/4 \end{pmatrix} \text{ (copy above)}$$

$$Ux = y \quad x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \text{ (copy above)}$$

- 2. Rearrange the equations  $Ax = b$  to form a strictly diagonally dominant system. Apply two steps of the Jacobi and Gauss-Seidel methods with starting vector  $[0, \dots, 0]$ :

$$A = \begin{pmatrix} 1 & -8 & -2 \\ 1 & 1 & 5 \\ 3 & -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

• **ans:**

Rearrange:

$$\begin{pmatrix} 3 & -1 & 1 \\ 1 & -8 & -2 \\ 1 & 1 & 5 \end{pmatrix} x = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

$$\begin{aligned} A &= L_t + D + U_t \\ &= \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 3 & & \\ & -8 & \\ & & 5 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & -1 & 1 \\ & 0 & -2 \\ & & 0 \end{pmatrix} \end{aligned}$$

Jacobi:

$$\begin{aligned} x_1 &= D^{-1}b - D^{-1}(L_t + U_t)x_0 \\ &= \begin{pmatrix} -2/3 \\ -1/8 \\ 4/5 \end{pmatrix} - \begin{pmatrix} 0 & -1/3 & 1/3 \\ -1/8 & 0 & 1/4 \\ 1/5 & 1/5 & 0 \end{pmatrix} x_0 \\ &= \begin{pmatrix} -2/3 \\ -1/8 \\ 4/5 \end{pmatrix} \\ x_2 &= D^{-1}b - D^{-1}(L_t + U_t)x_1 \\ &= \begin{pmatrix} -2/3 \\ -1/8 \\ 4/5 \end{pmatrix} - \begin{pmatrix} 0 & -1/3 & 1/3 \\ -1/8 & 0 & 1/4 \\ 1/5 & 1/5 & 0 \end{pmatrix} x_1 \\ &= \begin{pmatrix} -0.9750 \\ -0.4083 \\ 0.9583 \end{pmatrix} \end{aligned}$$

Gauss-Seidel:

$$(D + L_t)^{-1} = \begin{pmatrix} 1/3 & & \\ 1/24 & -1/8 & \\ -3/40 & 1/40 & 1/5 \end{pmatrix}$$

$$\begin{aligned} x_1 &= (D + L_t)^{-1}b - (D + L_t)^{-1}U_t x_0 \\ &= \begin{pmatrix} -0.6667 \\ -0.2083 \\ 0.9750 \end{pmatrix} - \begin{pmatrix} 0 & -0.3333 & 0.3333 \\ 0 & -0.0417 & 0.2917 \\ 0 & 0.0750 & -0.1250 \end{pmatrix} x_0 \\ &= \begin{pmatrix} -0.6667 \\ -0.2083 \\ 0.9750 \end{pmatrix} \\ x_2 &= (D + L_t)^{-1}b - (D + L_t)^{-1}U_t x_1 \\ &= \begin{pmatrix} -0.6667 \\ -0.2083 \\ 0.9750 \end{pmatrix} - (D + L_t)^{-1}U_t x_1 = \begin{pmatrix} -1.0611 \\ -0.5014 \\ 1.1125 \end{pmatrix} \end{aligned}$$

3. Find  $x_2$  by (a) Jacobi and by (b) Gauss-Seidel iterations, find (c)  $\|R_j\|_\infty$ , (d)  $\|R_{gs}\|_\infty$ , and (e) verify the error bounds

$$\|x - x_i\|_\infty \leq \|R_j\|_\infty^i \|x - x_0\|_\infty.$$

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

• **ans:**

(a) Jacobi:

$$\begin{aligned} x_1 &= D^{-1}(b - (L_t + U_t)x_0) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1/2 & 0 \end{pmatrix} x_0 \\ &= \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \\ x_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1/2 & 0 \end{pmatrix} x_1 \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

(b) Gauss-Seidel:

$$\begin{aligned} x_1 &= (L_t + D)^{-1}(b - U_t x_0) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} x_0 \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ x_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} x_1 \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

(c) Jacobi rate:

$$\begin{aligned} \|R_j\|_\infty &= \|D^{-1}(L_t + U_t)\|_\infty \\ &= \left\| \begin{pmatrix} 0 & 0 \\ -1/2 & 0 \end{pmatrix} \right\|_\infty \\ &= \max\{1/2, 0\} = 1/2 \end{aligned}$$

(d) Gauss-Seidel rate:

$$\begin{aligned} \|R_{gs}\|_\infty &= \|(L_t + D)^{-1}U_t\|_\infty \\ &= \max\{0, 0\} = 0 \end{aligned}$$

(e) Verify Jacobi error reduction bound:

Exact solution

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Errors

$$\|x - x_0\|_\infty = \left\| \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\|_\infty = 1$$

$$\|x - x_1\|_\infty = \left\| \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \right\|_\infty = 1/2$$

$$\|x - x_2\|_\infty = \left\| \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\|_\infty = 0$$

Is

$$\|x - x_i\|_\infty \leq \|R_j\|_\infty^i \|x - x_0\|_\infty?$$

$i = 1$ :

$$\frac{1}{2} \leq \frac{1}{2}(1)? \text{ Yes!}$$

$i = 2$ :

$$0 \leq \left(\frac{1}{2}\right)^2(1)? \text{ Yes!}$$

4. Show  $A$  is positive definite.

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}$$

• **ans:** If  $x \neq 0$ ,

$$\begin{aligned} x^T A x &= x_1^2 + 6x_1x_2 + 10x_2^2 \\ &= (x_1 + 3x_2)^2 + x_2^2 > 0 \end{aligned}$$

5. Show  $A$  is not positive definite.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

• **ans:**

$$x^T A x = x_1^2 + 4x_1x_2 + 2x_2^2 = (x_1 + x_2)^2 - 2x_2^2$$

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x^T A x = -2 < 0$$

6. Do CG iteration with  $x_0 = [0 \ 0]$ .

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$x_0 = 0; \quad d_0 = r_0 = b;$$

**for**  $i = 1 : n$ ; **if**  $r_i == 0$ , **break**; **end**

$$\alpha_i = r_i^T r_i / (d_i^T A d_i)$$

$$x_i = x_{i-1} + \alpha_i d_{i-1}; \quad r_i = r_{i-1} - \alpha_i A d_{i-1}$$

$$\beta_i = r_i^T r_i / (r_{i-1}^T r_{i-1}); \quad d_i = r_i + \beta_i d_{i-1}$$

**end**

• **ans:**

```
clc; b=[0 1]'; x=0*b; A=[1 -1; -1 2];
d=b, r=b,
```

```
al=(r'*r)/(d'*A*d), rats(al)
```

```
x = x + al * d
```

```
r1 = r - al * A * d
```

```
ba = (r1'*r1) / (r'*r)
```

```
d = r1 + ba * d
```

```
r=r1;
```

```
al=(r'*r)/(d'*A*d), rats(al)
```

```
x = x + al * d
```

```
r1 = r - al * A * d
```

$$d_0 = r_0 = x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

first iteration

$$\alpha_1 = \frac{r_0^T r_0}{d_0^T A d_0} = \frac{1}{2}$$

$$x_1 = x_0 + \alpha_1 d_0 = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

$$r_1 = r_0 - \alpha_1 A d_0 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$$\beta_1 = r_0^T r_0 / (r_1^T r_1) = \frac{1}{4}$$

$$d_1 = r_1 + \beta_1 d_0 = \begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix}$$

2nd iteration (at most 2 for 2 by 2 systems)

$$\alpha_2 = \frac{r_1^T r_1}{d_1^T A d_1} = 2$$

$$x_2 = x_1 + \alpha_2 d_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$r_2 = r_1 - \alpha_2 A d_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The exact solution is

$$x = x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

7. Find the  $P_3(x)$  interpolation by

(1) solving equations for unknown coefficients,

(2) Lagrange nodal basis,

(3) Newton's divided differences.

$$\begin{array}{c|c|c|c|c} x_i & -1 & 0 & 1 & 2 \\ \hline y_i & 0 & -2 & 0 & 0 \end{array}$$

• **ans:** Method(1) We look for a degree 3 polynomial to fit 4 data points.

$$P_3(x) = a + bx + cx^2 + dx^3$$

For example, when  $x_0 = -1$ ,  $y_0 = 0$  we get one equation:

$$0 = a - b + c - d$$

Then for the unknown coefficients (vector  $X = (a, b, \dots)$ ), we get the following equations obtained by evaluating the polynomial at the given points:

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

Solve the system of equations:

$$X = \begin{pmatrix} -2 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

$$P_3(x) = -2 + x + 2x^2 - x^3$$

Method(2) Using Lagrange basis

$$\begin{aligned} p_3(x) &= y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \\ &+ y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ &+ y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \\ &+ y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\ &= 0 - 2 \frac{(x+1)(x-1)(x-2)}{(1)(-1)(-2)} + 0 + 0 \\ &= -(x^2-1)(x-2) \\ &= -x^3 + 2x^2 + x - 2 \end{aligned}$$

We must get the same answer. Also we can check the function by evaluate it at the 4 given point!

Method (3) Newton's divided differences:

$$\begin{aligned} f[x_k] &= f(x_k) \\ f[x_k, x_{k+1}] &= \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k} \\ f[x_k, x_{k+1}, x_{k+2}] &= \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k} \\ &= \frac{f[x_k, x_{k+1}, x_{k+2}, x_{k+3}] - f[x_{k+1}, x_{k+2}, x_{k+3}]}{x_{k+3} - x_k} \end{aligned}$$

$$P = \sum_{i=0}^n f[x_0 \dots x_i] (x-x_0) \dots (x-x_{i-1})$$

We can use the above formulas directly. But it is much easier to use the following table.

$$\begin{pmatrix} -1 & \boxed{0} & & & \\ & & \boxed{-2} & & \\ 0 & -2 & & \boxed{2} & \\ & & 2 & & \boxed{-1} \\ 1 & 0 & & -1 & \\ & & 0 & & \\ 2 & 0 & & & \end{pmatrix}$$

$$\begin{aligned} P_3 &= 0 - 2(x+1) + 2(x+1)(x) - (x+1)x(x-1) \\ &= -2x - 2 + 2x^2 + 2x - x^3 + x \\ &= -x^3 + 2x^2 + x - 2 \end{aligned}$$

8. Find  $P_9(0)$  interpolating  $f(x)$  at the given points, if  $f(1) = 112$ ,  $f(10) = 2$  and  $f(i) = 0$ ,  $i = 2, 3, 4, 5, 6, 7, 8, 9$ .

• **ans:** 2 Methods.

(1) using Lagrange basis:

$$\begin{aligned} P_9(x) &= f(1) \frac{(x-2)(x-3)\dots(x-9)(x-10)}{(1-2)(1-3)\dots(1-9)(1-10)} + 0 + \dots \\ &+ f(10) \frac{(x-1)(x-2)\dots(x-9)}{(10-1)(1-2)\dots(10-9)} \\ &= \dots \end{aligned}$$

But the second method is better:

(2) Solving equations:

$$\begin{aligned} P_9(x) &= (A+Bx)(x-2)(x-3)(x-4)(x-5) \\ &\quad (x-6)(x-7)(x-8)(x-9) \\ 112 &= f(1) = (A+B)(8!) \\ 2 &= f(10) = (A+10B)(8!) \end{aligned}$$

Equation 2 minus equation 1:

$$-110 = 9B(8!), \quad B = -\frac{110}{9!}$$

Equation 2 minus 10 times equation 1:

$$2 - 1120 = -9A(8!), \quad A = -\frac{1118}{9!}$$

$$P_9(0) = A(-9!) = 1118$$

9. Let  $f(x) = \ln x$ . Let  $x_i = 1, 2, 4$

- Interpolate  $f$  at the three points by a  $P_2$ .
- Find  $f(3)$ ,  $P_2(3)$ , and the error.
- Find the bound on error  $e(x) = f(x) - P_2(x)$  by the Lagrange theory.
- Compare error bound with actual error  $e(3)$

• **ans:**

- We look for a degree 3 polynomial to fit 3 data points.

$$P_2(x) = A + Bx + Cx^2$$

Then for the unknown coefficients (vector  $X = (A, B, \dots)$ ), we get the following equations obtained by evaluating the polynomial at the given points:

$$\begin{pmatrix} 1. & 1 & 1 \\ 1. & 2 & 2^2 \\ 1. & 4 & 4^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ \ln 2 \\ \ln 4 \end{pmatrix}$$

Note that the rhs are values  $f(x_i)$

$$X = \begin{pmatrix} -0.9242 \\ 1.0397 \\ -0.1155 \end{pmatrix}$$

$$P_2(x) = -0.9242 + 1.0397x - 0.1155x^2$$

(b)

$$e(3) = f(3) - P_2(3) = -0.0568$$

(c)

$$\begin{aligned} e(x) &= \frac{f'''(z)}{3!}(x-x_0)(x-x_1)(x-x_2) \\ f'''(z) &= 2z^{-3} \\ e(x) &= \frac{2z^{-3}}{6}(x-1)(x-2)(x-4) \\ |e(3)| &= \left| \frac{2z^{-3}}{6}(-2) \right| \\ &\leq \frac{2(1^{-3})}{6}(2) = \frac{2}{3} = 0.66 \\ |e(3)| &= 0.0568 \text{ yes, smaller} \end{aligned}$$

10. Find the Bezier curve  $P_3(t) = \mathbf{a} + \mathbf{b}t + \mathbf{c}t^2 + \mathbf{d}t^3$ , given 4 points:

$$\mathbf{x}_i = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

$$\begin{aligned} \mathbf{a} &= \mathbf{x}_1; & \mathbf{b} &= 3(\mathbf{x}_2 - \mathbf{x}_1); \\ \mathbf{c} &= 3(\mathbf{x}_3 - \mathbf{x}_2) - \mathbf{b}; & \mathbf{d} &= \mathbf{x}_4 - \mathbf{x}_1 - \mathbf{b} - \mathbf{c}. \end{aligned}$$

• **ans:**

$$\mathbf{a} = \mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\mathbf{b} = 3(\mathbf{x}_2 - \mathbf{x}_1) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\mathbf{c} = 3(\mathbf{x}_3 - \mathbf{x}_2) - \mathbf{b} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

$$\mathbf{d} = \mathbf{x}_4 - \mathbf{x}_1 - \mathbf{b} - \mathbf{c} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

$$\mathbf{P}_3(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}t + \begin{pmatrix} 3 \\ -3 \end{pmatrix}t^2 + \begin{pmatrix} -2 \\ 0 \end{pmatrix}t^3$$

or

$$\begin{aligned} x(t) &= 1 + 3t^2 - 2t^3 \\ y(t) &= 2 + 3t - 3t^2 \end{aligned}$$

check:

$$\begin{aligned} \mathbf{P}(0) &= \mathbf{x}_1, \quad \mathbf{P}'(0) = 3(\mathbf{x}_2 - \mathbf{x}_1), \\ \mathbf{P}(1) &= \mathbf{x}_4, \quad \mathbf{P}'(1) = 3(\mathbf{x}_4 - \mathbf{x}_3) \end{aligned}$$

11. Find the  $\mathbf{x}_i$  for the Bezier Curve

$$\begin{aligned} x(t) &= 2 + t^2 - t^3; & y(t) &= 1 - t + 2t^3. \\ \mathbf{x}_1 &= \mathbf{a}; & \mathbf{x}_2 &= \mathbf{b}/3 + \mathbf{x}_1; \\ \mathbf{x}_3 &= \mathbf{b}/3 + \mathbf{c}/3 + \mathbf{x}_2; & \mathbf{x}_4 &= \mathbf{d} + \mathbf{x}_1 + \mathbf{b} + \mathbf{c}. \end{aligned}$$

• **ans:** Method 1:

$$\begin{aligned} \mathbf{x}_1 \mathbf{P}(0), \quad \mathbf{x}_2 &= \mathbf{P}'(0)/3 + \mathbf{x}_1, \\ \mathbf{x}_4 &= \mathbf{P}(1), \quad \mathbf{x}_3 = \mathbf{x}_4/\mathbf{P}'(1) \end{aligned}$$

Method 2:

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \mathbf{x}_2 &= \mathbf{b}/3 + \mathbf{x}_1 = \begin{pmatrix} 2 & 2/3 \end{pmatrix} \\ \mathbf{x}_3 &= \mathbf{b}/3 + \mathbf{c}/3 + \mathbf{x}_2 = \begin{pmatrix} 7/3 \\ 1/3 \end{pmatrix} \\ \mathbf{x}_4 &= \mathbf{d} + \mathbf{x}_1 + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{aligned}$$

12. Find the least-squares solution for the over-determined system. Compare the residual for the least-squares solution with “solutions” (1) (0, 0), (2) (1, 1).

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix} x = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

• **ans:**

$$A^T A x = A^T b$$

$$A^T A = \begin{pmatrix} 6 & -5 \\ -5 & 10 \end{pmatrix}$$

$$(A^T A)^{-1} = \begin{pmatrix} 0.28571 & 0.14286 \\ 0.14286 & 0.17143 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$x = (A^T A)^{-1} A^T b = \begin{pmatrix} 0.2857 \\ -0.2571 \end{pmatrix}$$

$$r = b - Ax = \begin{pmatrix} -1.0286 \\ 1.7143 \\ -0.3429 \end{pmatrix},$$

$$\|b - Ax\|_2 = \sqrt{1.0286^2 + \dots} = 2.0284$$

$$r_1 = b - A \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$\|r_1\|_2 = 2.4495 \text{ bigger}$$

$$r_2 = b - A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$$\|r_2\|_2 = 3.7417 \text{ bigger}$$

13. Find the least-squares solution for the under-determined system. Compare the length for the least-squares solution with solutions (1)  $(14, 0, 28)$ , (2)  $(0, 70/3, 70/3)$ , (3) a solution found by Gauss elimination.

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 70 \\ -70 \end{pmatrix}.$$

• **ans:**

$$yA^T x$$

$$AA^T y = b$$

$$AA^T = \begin{pmatrix} 6 & -5 \\ -5 & 10 \end{pmatrix}$$

$$(AA^T)^{-1} = \begin{pmatrix} 0.28571 & 0.14286 \\ 0.14286 & 0.17143 \end{pmatrix}$$

$$y = (AA^T)^{-1} b = \begin{pmatrix} 10 \\ -2 \end{pmatrix}$$

$$x = A^T y = \begin{pmatrix} 8 \\ 10 \\ 26 \end{pmatrix}$$

$$r = b - Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ (checking)}$$

$$\|x\|_2 = \sqrt{8^2 + 10^2 + 26^2} = 28.98$$

- (1) Solution  $\mathbf{x}_1 = (14, 0, 28)$

$$r_1 = b - A \begin{pmatrix} 14 \\ 0 \\ 28 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ (checking)}$$

$$\|\mathbf{x}_1\|_2 = \sqrt{14^2 + 0^2 + 28^2} = 31.3 \text{ bigger}$$

- (2) Solution  $\mathbf{x}_2 = (0, 70/3, 70/3)$

$$r_2 = b - A \begin{pmatrix} 0 \\ 70/3 \\ 70/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ (checking)}$$

$$\|\mathbf{x}_2\|_2 = 32.99 \text{ bigger}$$

- (3) Gaussian elimination:

$$\left( \begin{array}{ccc|c} 1 & 1 & 2 & 70 \\ 1 & 0 & -3 & -70 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 2 & 70 \\ & -1 & -5 & -140 \end{array} \right)$$

$x_3$  is free. Trying to make  $\|x\|$  small, we let  $x_3 = 0$ . Then

$$x = \begin{pmatrix} -70 \\ 140 \\ 0 \end{pmatrix}$$

Check:

$$b - Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\|x\|_2 = \sqrt{70^2 + 140^2 + 0} = 70\sqrt{5} = 156.5$$

Much bigger than that of the least square solution.

14. Find the least-squares solution each problem.

$$\begin{pmatrix} -2 & -1 \\ 1 & 1 \\ 3 & -1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 0 \\ 14 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 & 3 \\ -1 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 21 \\ 9 \end{pmatrix}.$$

• **ans:**

$$A^T A x = A^T b$$

$$A^T A = \begin{pmatrix} 14 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 28 \\ -21 \end{pmatrix}$$

$$x = (A^T A)^{-1} A^T b = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$$

$$AA^T y = b \Rightarrow y = \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$$

$$x = A^T y = \begin{pmatrix} -6 \\ 4.5 \\ 1.5 \end{pmatrix}$$

15. Approximate the function  $z = f(x, y)$  by a plane.

$x$	0	0	1	1	1
$y$	0	1	0	1	2
$f(x, y)$	3	2	3	5	6

• **ans:**  $z = a + bx + cy$ . Plug in all data points:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 5 \\ 6 \end{pmatrix}$$

$$A^T A \mathbf{x} = A^T \mathbf{b}$$

$$\begin{pmatrix} 5 & 3 & 4 \\ 3 & 3 & 3 \\ 4 & 3 & 6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 19 \\ 14 \\ 19 \end{pmatrix}$$

$$\mathbf{x} = \frac{1}{3} \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix}$$

$$z = 2 + \frac{5}{3}x + y.$$

16. Let  $f(x) = x + \sin x$ ,  $x_0 = 0.5$ .

- (a) Approximate  $f'(x_0)$  by the (3-point) central differences with  $h = 0.1$  and  $h = 0.2$ . Find the error bound for  $h = 0.1$ . Find the Richardson extrapolation. Check all three errors.
- (b) Using the (3-point) central differences with  $h = 0.1$  and  $h = 0.2$ , Find the error bound for  $h = 0.1$ . Find the extrapolation to approximate  $f''(x_0)$ . Check all three errors.

$$\begin{aligned} f_0''(h = 0.2) &= -0.4778 \\ f''(x_0) - f_0'' &= -0.0016 \\ f_0'' &= \frac{4f_0''(h = 0.1) - f_0''(h = 0.2)}{4 - 1} \\ &= -0.4794 \\ f''(x_0) - f_0'' &= -0.00000053 \end{aligned}$$

• **ans:**

(a)

$$\begin{aligned} f_0' &= \frac{f_1 - f_{-1}}{2h} \\ f_0' &= \frac{f(x_0 + h) - f(x_0 - h)}{2h} \\ &= 1.8761 \end{aligned}$$

$$e = f'(x_0) - f_0' = 0.0015$$

Error bound:

$$f''' = -\cos(x)$$

$$\begin{aligned} |f'(x_0) - f_0'| &\leq \frac{h^2}{6} \max_{x_{-1} \leq z \leq x_1} |f'''(z)| \\ &\leq \frac{0.1^2}{6} (\cos(0.5 - 0.1)) = 0.0015 \end{aligned}$$

$$f_0'(h = 0.1) = 1.8761$$

$$f_0'(h = 0.2) = 1.8717$$

$$e(h = 0.2) = 0.0058$$

$$\begin{aligned} f_0' &= \frac{4f_0'(h = 0.1) - f_0'(h = 0.2)}{4 - 1} \\ &= 1.8776 \end{aligned}$$

$$\text{new error} = f'(x_0) - f_0' = 0.0000029$$

The new error must be much smaller!

(b)

$$\begin{aligned} f_0'' &= \frac{f_1 - 2f_0 + f_{-1}}{h^2} \\ f_0''(h = 0.1) &= \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} \\ &= -0.4790 \\ f''(x_0) - f_0'' &= -0.0004 \end{aligned}$$

Error bound:

$$f'''' = \sin(x)$$

$$\begin{aligned} |f''(x_0) - f_0''| &\leq \frac{h^2}{12} \max_{x_{-1} \leq z \leq x_1} |f''''(z)| \\ &\leq \frac{0.1^2}{12} (\sin(0.5 + 0.1)) = 0.00047 \end{aligned}$$