M353 Study Guide 2 (S. Zhang) .

1. Solve the following system $Ax = b$

   (a) by GE without pivoting, (no row switching, no row multiplication)

   (b) by finding $A = LU$ and using it

   (c) by GE with partial pivoting, (row pivoting every step)

   (d) by finding $PA = LU$ and using it

$$(A|b) = \begin{pmatrix} 1 & -3 & 2 & | & 5 \\ -2 & 0 & 2 & | & 2 \\ 3 & -1 & -1 & | & 0 \end{pmatrix}$$

2. Rearrange the equations $Ax = b$ to form a strictly diagonally dominant system. Apply two steps of the Jacobi and Gauss-Seidel methods with starting vector $[0, .., 0]$:

$$A = \begin{pmatrix} 1 & -8 & -2 \\ 1 & 1 & 5 \\ 3 & -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

3. Find x_2 by (a) Jacobi and by (b) Gauss–Seidel iterations, find (c) $\|R_j\|_\infty$, (d) $\|R_{gs}\|_\infty$, and (e) verify the error bounds $\|x - x_i\|_\infty \le \|R_j\|_\infty^i \|x - x_0\|_\infty$.

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

4. Show A is positive definite.

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}$$

5. Show A is not positive definite.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

6. Do CG iteration with $x_0 = [0\ 0]$.

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$x_0 = 0;\ d_0 = r_0 = b;$

for $i = 1 : n;$ **if** $r_i == 0,$ **break**; **end**

$\quad \alpha_i = r_{i-1}^T r_{i-1} / (d_{i-1}^T A d_{i-1})$

$\quad x_i = x_{i-1} + \alpha_i d_{i-1}; \quad r_i = r_{i-1} - \alpha_i A d_{i-1}$

$\quad \beta_i = r_i^T r_i / (r_{i-1}^T r_{i-1}); \quad d_i = r_i + \beta_i d_{i-1}$

end

7. Find the $P_3(x)$ interpolation by
   (1) solving equations for unknown coefficients,
   (2) Lagrange nodal basis,
   (3) Newton's divided differences.

x_i	-1	0	1	2
y_i	0	-2	0	0

8. Find $P_9(0)$ interpolating $f(x)$ at the given points, if $f(1) = 112$, $f(10) = 2$ and $f(i) = 0$, $i = 2, 3, 4, 5, 6, 7, 8, 9$.

9. Let $f(x) = \ln x$. Let $x_i = 1, 2, 4$

   (a) Interpolate f at the three points by a P_2.

   (b) Find $f(3)$, $P_2(3)$, and the error .

   (c) Find the bound on error $e(x) = f(x) - P_2(x)$ by the Lagrange theory.

   (d) Compare error bound with actual error $e(3)$

10. Find the Bezier curve $P_3(t) = \mathbf{a} + \mathbf{b}t + \mathbf{c}t^2 + \mathbf{d}t^3$, given 4 points:

$$\mathbf{x}_i = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

$\mathbf{a} = \mathbf{x}_1;$ $\quad \mathbf{b} = 3(\mathbf{x}_2 - \mathbf{x}_1);$

$\mathbf{c} = 3(\mathbf{x}_3 - \mathbf{x}_2) - \mathbf{b};$ $\quad \mathbf{d} = \mathbf{x}_4 - \mathbf{x}_1 - \mathbf{b} - \mathbf{c}.$

11. Find the $\mathbf{x}_i$ for the Bezier Curve

$x(t) = 2 + t^2 - t^3;$ $\quad y(t) = 1 - t + 2t^3.$

$\mathbf{x}_1 = \mathbf{a};$ $\quad \mathbf{x}_2 = \mathbf{b}/3 + \mathbf{x}_1;$

$\mathbf{x}_3 = \mathbf{b}/3 + \mathbf{c}/3 + \mathbf{x}_2;$ $\quad \mathbf{x}_4 = \mathbf{d} + \mathbf{x}_1 + \mathbf{b} + \mathbf{c}.$

12. Find the least-squares solution for the over-determined system. Compare the residual for the least-squares solution with "solutions" (1) $(0, 0)$, (2) $(1, 1)$.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix} x = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

13. Find the least-squares solution for the under-determined system. Compare the length for the least-squares solution with solutions (1) $(14, 0, 28)$, (2) $(0, 70/3, 70/3)$, (3) a solution found by Gauss elimination.

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 70 \\ -70 \end{pmatrix}.$$

14. Find the least-squares solution each problem.

$$\begin{pmatrix} -2 & -1 \\ 1 & 1 \\ 3 & -1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 0 \\ 14 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 & 3 \\ -1 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 21 \\ 9 \end{pmatrix}.$$

15. Approximate the function $z = f(x, y)$ by a plane.

x	0	0	1	1	1
y	0	1	0	1	2
$f(x, y)$	3	2	3	5	6

16. Let $f(x) = x + \sin x$, $x_0 = 0.5$.

   (a) Approximate $f'(x_0)$ by the (3-point) central differences with $h = 0.1$ and $h = 0.2$. Find the error bound for $h = 0.1$. Find the Richardson extrapolation. Check all three errors.

(b) Using the (3-point) central differences with $h = 0.1$ and $h = 0.2$, Find the error bound for $h = 0.1$. Find the extrapolation to approximate $f''(x_0)$. Check all three errors.