M353 Study Guide 1 (S. Zhang) .

1. Find the following sums by hand in IEEE double precision computer arithmetic, using the Rounding to Nearest Rule:

$$(a)(1+(2^{-51}+2^{-52}+2^{-54}))-1$$
$$(b)(1+(2^{-51}+2^{-52}+2^{-60}))-1$$

- **ans:** (a)

$$\begin{aligned}(1+(2^{-51}+2^{-52}+2^{-54}))-1 &= (1+(2^{-51}+2^{-52}))-1\\ &= 2^{-51}+2^{-52}\end{aligned}$$

We do it again with details.

$$2^{-51}+2^{-52}+2^{-54} = 1.101_2 \times 2^{-51}$$

$$p=-51=-(32+16+0(8)+0(4)+2+1)=-110011_2$$

$$11111111111_2 - 110011_2 = 11110\,01100_2$$

$$\boxed{0}\ \boxed{0\,11110\,01100}\ \underset{52\text{ bit}}{\boxed{1010...0}}$$

$$\begin{aligned}&1+(2^{-51}+2^{-52}+2^{-54})\\ &=1.00...001101_2\\ &\simeq 1.\underset{52\text{ bit}}{\boxed{00..0011}}\times 2^0\end{aligned}$$

$$\boxed{0}\ \boxed{0\,11111\,11111}\ \underset{52\text{ bit}}{\boxed{00..0011}}$$

$$\begin{aligned}&(1+(2^{-51}+2^{-52}+2^{-54})-1)\\ &\simeq 1.00...0011_2 - 1 = 1.1_2\times 2^{-51}\end{aligned}$$

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Matlab:
 x=(1+(2^-51 +2^-52 +2^-54))-1
 x*2^52
answer: 6.6613e-16  3
 x=(1+(2^-51 +2^-52 +2^-60))-1
 x*2^52
answer: 6.6613e-16  3
```

(b)

$$\begin{aligned}(1+(2^{-51}+2^{-52}+2^{-60}))-1 &= (1+(2^{-51}+2^{-52}))-1\\ &= 2^{-51}+2^{-52}\end{aligned}$$

We do it again with details.

$$2^{-51}+2^{-52}+2^{-60} = 1.1000000001_2 \times 2^{-51}$$

$$p=-51=-(32+16+0(8)+0(4)+2+1)=-110011_2$$

$$11111111111_2 - 110011_2 = 11110\,01100_2$$

$$\boxed{0}\ \boxed{0\,11110\,01100}\ \underset{52\text{ bit}}{\boxed{1000000010...0}}$$

$$\begin{aligned}&1+(2^{-51}+2^{-52}+2^{-54})\\ &=1.00...0011000000001_2\\ &\simeq 1.\underset{52\text{ bit}}{\boxed{00..0011}}\times 2^0\end{aligned}$$

$$\boxed{0}\ \boxed{0\,11111\,11111}\ \underset{52\text{ bit}}{\boxed{00..0011}}$$

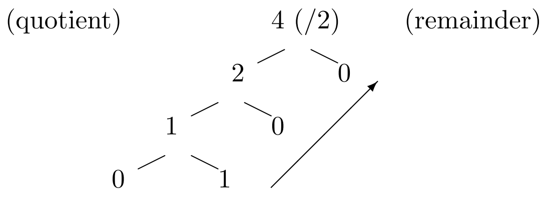
$$\begin{aligned}&(1+(2^{-51}+2^{-52}+2^{-64})-1)\\ &\simeq 1.00...0011_2 - 1 = 1.1_2\times 2^{-51}\end{aligned}$$

2. Convert $a=4.125$ and $b=19/7$ to IEEE doubles. Then find the IEEE double form for $a+b$ (using chopping after bit 52.)

- **ans:**

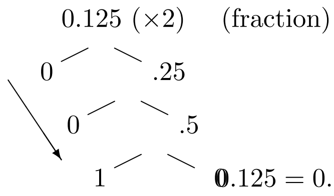
$$4.125 = 4+0.124$$

Convert 4



$$4 = 100_2$$

Convert 0.125.



$0.125 = 0.001_2$,

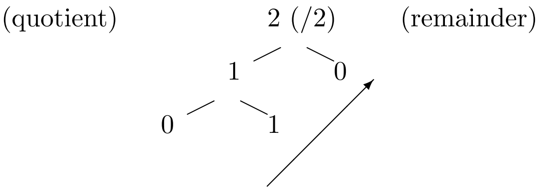
$$4.125 = 100.001_2 = 1.00001_2\times 2^2$$

Use IEEE double: $p = 2$. $1111111111_2 + 2 = 1\,00000\,00001_2$

$$\boxed{0}\ \boxed{1\,00000\,00001}\ \underset{\text{52 bit}}{\boxed{000010...0}}$$

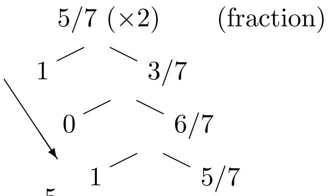
$$19/7 = 2 + \frac{5}{7}$$

Convert 2



$$2 = 10_2$$

Convert 5/7.



$$\frac{5}{7} = 0.\overline{101}_2$$

$$\frac{19}{7} = 10.\overline{101}_2 = 1.0\overline{101}_2 \times 2^1$$

Use IEEE double: $p = 1$. $1111111111_2 + 1 = 1\,00000\,00000_2$

$$\boxed{0}\ \boxed{1\,00000\,00000}\ \underset{\text{52 bit}}{\boxed{0\,101\,101\,...101}}$$

$$\begin{aligned}
&4.125 + \frac{19}{7} \\
=&1.00001_2 \times 2^2 + 1.0\overline{101}_2 \times 2^1 \\
=&1.\underset{\text{52 bit}}{\boxed{00\,001\,000\,...000\,00}} \times 2^2 \\
+&0.\underset{\text{52 bit}}{\boxed{10\,101\,101\,...101\,10}} \times 2^2 \\
=&1.\underset{\text{52 bit}}{\boxed{10\,110\,101\,...101\,10}} \times 2^2
\end{aligned}$$

We can convert the answer back to decimal, but not required.

3. Find an interval of length one that contains a root.

$$x^4 = x^2 + 2$$

Then do two steps of bisection method. How many steps of bisection iterations are need to reach 10^{-10} accuracy?

- **ans:**

$$f(x) = x^4 - x^2 - 2$$

$$f(0) = -2,\ f(1) = -2,\ f(2) = 10$$

$$[a, b] = [1, 2]$$

After we compute the first c, we will replace either a or b by c so that the new interval can still trap a root inside it.

$$c = \frac{a+b}{2}$$

i	$a, f(a)$	$c, f(c)$	$b, f(b)$
0	$1, [-2]$	$1.5, [0.8125]$	$2, [10]$
1	$1, [-2]$	$1.25, [-1.1211]$	$1.5, [0.8125]$
2	$1.25, [-1.1211]$		$1.5, [0.8125]$

$$x = 1.375$$

Error bound:

$$|r - c_n| \le \frac{1}{2^{n+1}}(b - a)$$

$$\frac{1}{2^{n+1}}(2 - 1) \le 10^{-10}$$

$$(n+1)\ln\frac{1}{2} \le \ln 10^{-10}$$

$$(n+1) \ge \frac{-10\ln 10}{\ln\frac{1}{2}} = 33.21$$

$$n \ge 33.$$

4. Express the equation as a fixed point iteration in 3 ways.

$$x^4 + 9x^3 = x^2$$

Find the fixed point iteration for the Newton's method.

- **ans:** Method 1.

$$\begin{aligned}
x^4 &= -9x^3 + x^2 \\
x &= \left(-9x^3 + x^2\right)^{1/4} = g(x)
\end{aligned}$$

Method 2.

$$\begin{aligned}
9x^3 &= -x^4 + x^2 \\
x &= \left(-\frac{1}{9}x^4 + \frac{1}{9}x^2\right)^{1/3} = g(x)
\end{aligned}$$

Method 3.

$$x^2 = x^4 + 9x^3$$

$$x = \left(x^4 + 9x^3\right)^{1/2} = g(x)$$

Method 4. Newton's method.

$$\begin{aligned} f(x) &= x^4 + 9x^3 - x^2 \\ f'(x) &= 4x^3 + 27x^2 - 2x \end{aligned}$$

$$\begin{aligned} x^4 + 9x^3 &= x^2 \\ 0 &= -x^4 - 9x^3 + x^2 = -f(x) \text{ [move 2 terms]} \\ 0 &= -\frac{x^4 + 9x^3 - x^2}{4x^3 + 27x^2 - 2x} \text{ [divided by } f'(x)\text{]} \\ x &= x - \frac{x^4 + 9x^3 - x^2}{4x^3 + 27x^2 - 2x} = g(x) \text{ [add } x\text{].} \end{aligned}$$

5.

$$g(x) = -4 + 4x - \frac{1}{2}x^2$$

(a) Find fixed points.
(b) Find the local convergence at each fixed point.
(c) Do 2 steps of fixed point iteration with $p_0 = 1.9$.
(d) Do 2 steps of fixed point iteration with $p_0 = 3.8$.
(e) Find errors above.
(f) What about the convergence?

- **ans:**

(a)

$$x = g(x) = -4 + 4x - \frac{1}{2}x^2$$

$$\frac{1}{2}x^2 - 3x + 4 = 0$$

$$(\frac{1}{2}x - 2)(x - 2) = 0$$

$$x = 4, 2.$$

(b)

$$g'(x) = 4 - x$$

$$S = |g'(4)| = 0 < 1$$

So loclly convergent.

$$S = |g'(2)| = 2 > 1$$

So FPI is not loclly convergent.

(c)

$$\begin{aligned} p_1 &= g(p_0) = 1.7950 \\ p_2 &= g(p_1) = 1.5689875 \\ p_3 &= g(p_2) = 1.045089112421 \end{aligned}$$

The iteration diverges. In fact,

$$\begin{aligned} p_4 &= -0.365749176763876 \\ p_5 &= -5.52988293720723 \end{aligned}$$

(d)

$$\begin{aligned} p_1 &= g(p_0) = 3.980 \\ p_2 &= g(p_1) = 3.9998 \\ p_3 &= g(p_2) = 3.99999998 \end{aligned}$$

The iteration converges to 4.

(e)

$$\begin{aligned} E_0 &= 2 - p_0 = 0.1 \\ E_1 &= 2 - p_1 = 0.205 \\ E_2 &= 2 - p_2 = 0.4310125 \\ E_3 &= 2 - p_3 = 0.95491 \end{aligned}$$

$$\begin{aligned} E_0 &= 4 - p_0 = 0.2 \\ E_1 &= 4 - p_1 = 0.02 \\ E_2 &= 4 - p_2 = 0.0002 \\ E_3 &= 4 - p_3 = 0.00000002 \end{aligned}$$

(f) What about the convergence? For $p_0 = 1.9$. It diverges. In fact, near the fixed point $p = 2$,

$$g'(x) = 4 - x \sim 2 > 1$$

So the fixed point theorem fails.

For $p_0 = 1.9$. It diverges. In fact, near the fixed point $p = 4$,

$$g'(x) = 4 - x \sim 0 << 1$$

So the fixed point theorem guarantees the convergence of the iteration if p_0 is close to $p = 4$ enough.

6. Find convergence order and the rate of the Newton's method

$$32x^3 - 32x^2 - 6x + 9 = 0, \; r = -1/2, 3/4$$

- **ans:**

$$\begin{aligned} f'(x) &= 96x^2 - 64x - 6 \\ f''(x) &= 192x - 64 \end{aligned}$$

$$\begin{aligned} r &= -1/2 \\ f'(r) &= 50 \neq 0 \\ f''(r) &= -160 \\ \frac{e_{i+1}}{e_i^2} &\to M = \left|\frac{f''(r)}{2f'(r)}\right| = \frac{8}{5} \end{aligned}$$

We have a quadratic convergence.

$$\begin{aligned} r &= 3/4 \\ f'(r) &= 0 \\ f''(r) &= 80 \neq 0 \\ \frac{e_{i+1}}{e_i} &\to S = \frac{m-1}{m} = \frac{2-1}{2} = \frac{1}{2} \end{aligned}$$

We have a linear convergence.

7. For finding the root of the function:

$$f(x) = x^2 - 5x - 11$$

   (a) Do 3 steps of the Newton's method, $p_0 = 6$. Find the errors and use the data to show the method is a second-order one.
   (b) Do 3 steps of the secant method, $p_0 = 0$, $p_1 = 8$.
   (c) Do 3 steps of the false position method, given the initial interval $[0, 8]$.

- **ans:**

(a) Each iteration would double the number of correct digits, for a second order iteration. Exact solution

$$x = 6.6533$$

$$\begin{aligned} p_1 &= p_0 - f(p_0)/f'(p_0) \\ f'(x) &= 2x - 5 \end{aligned}$$

k	p	$f(p)$	$f'(p)$	error
1	6.0000	−5.0000	7.0000	0.6533
2	6.7143	0.5102	8.4286	−0.0610
3	6.6538	0.0037	8.3075	−0.0004

The errors are reduced by 1/10 and 1/100. if we do one more iteration, the error is to be reduced by 1/10000. If we do a few more Newton's iterations, we get the exact solution

$$x = 6.65331173.$$

(b)

$$p_2 = p_1 - f(p_1)\frac{p_1 - p_0}{f(p_1) - f(p_0)}$$

k	p	$f(p)$
0	0.0000	−11.0000
1	8.0000	13.0000
2	3.6667	−15.8889
3	6.0500	−4.6475
4	7.0353	3.3193

(c)

$$c = b - f(b)\frac{b-a}{f(b)-f(a)}$$

i	$a, f(a)$	$c, f(c)$	$b, f(b)$
1	$0.0000_{-11.0000}$	$3.6667_{-15.8889}$	$8.0000_{13.0000}$
2	$3.6667_{-15.8889}$	$6.0500_{-4.6475}$	$8.0000_{13.0000}$
3	$6.0500_{-4.6475}$	$6.5635_{-0.7377}$	$8.0000_{13.0000}$

8. Find $\det(A)$ by expansions directly. Then again, by reducing A to an upper triangular matrix.

$$A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 2 & -1 \\ 2 & 2 & 5 & 0 \end{pmatrix}$$

- **ans:** Expand by column 1:

$$\begin{aligned} \det A &= (1)(-1)^{1+1} \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & -1 \\ 2 & 5 & 0 \end{pmatrix} \\ &\quad + 0 + 0 + (2)(-1)^{4+1} \det \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{pmatrix} \\ &= (-5) - 2(0) = -5 \end{aligned}$$

$$\begin{aligned} A &\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2.5 \end{pmatrix} \end{aligned}$$

$$\det A = (1)(1)(-2)(2.5) = -5$$

9. Find A^{-1} by elementary row operations,

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & -1 & 3 & 3 \\ -1 & 0 & -2 & 0 \\ 0 & 3 & -4 & 6 \end{pmatrix}$$

- **ans:**

$$[A|I] = \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & -1 & 3 & 3 & 0 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & -4 & 6 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-2r_1+r_2} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ & -1 & 1 & -1 & -2 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & -4 & 6 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_1+r_3} \left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & & \\ & -3 & 4 & 1 & 1 & \\ & 2 & -3 & -1 & & 1 \end{array}\right)$$

$$\xrightarrow{(-1/3)r_2} \left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & & \\ & 1 & -4/3 & -1/3 & -1/3 & \\ & 2 & -3 & -1 & & 1 \end{array}\right)$$

$$\xrightarrow{3r_2+r_1} \left(\begin{array}{ccc|ccc} 1 & & -2 & 0 & -1 & \\ & 1 & -4/3 & -1/3 & -1/3 & \\ & 2 & -3 & -1 & & 1 \end{array}\right)$$

$$\xrightarrow{-2r_2+r_3} \left(\begin{array}{ccc|ccc} 1 & & -2 & 0 & -1 & \\ & 1 & -4/3 & -1/3 & -1/3 & \\ & & -1/3 & -1/3 & 2/3 & 1 \end{array}\right)$$

$$\xrightarrow{(-3)r_3} \left(\begin{array}{ccc|ccc} 1 & & -2 & 0 & -1 & \\ & 1 & -4/3 & -1/3 & -1/3 & \\ & & 1 & 1 & -2 & -3 \end{array}\right)$$

$$\xrightarrow{(4/3)r_3+r_2} \left(\begin{array}{ccc|ccc} 1 & & -2 & 0 & -1 & \\ & 1 & & 1 & -3 & -4 \\ & & 1 & 1 & -2 & -3 \end{array}\right)$$

$$\xrightarrow{2r_3+r_1} \left(\begin{array}{ccc|ccc} 1 & & -2 & 2 & -5 & -6 \\ & 1 & & 1 & -3 & -4 \\ & & 1 & 1 & -2 & -3 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 2 & -5 & -6 \\ 1 & -3 & -4 \\ 1 & -2 & -3 \end{pmatrix}$$

10. Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 4 \\ -6 & 1 & 1.5 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 8 \\ -4.5 \end{pmatrix}.$$

(a) Use Gaussian elimination without pivoting to solve $Ax = b$.

(b) Find $A = LU$ where L is the unit lower triangular matrix.

(c) Using LU decomposition of A to solve $Ax = b$.

(d) Use Gaussian elimination with partial pivoting to solve $Ax = b$.

- **ans:**

(a) All intermediate results must be the same too! Make sure that the rule of Gaussian elimination is understood and you do not use any "smart steps".

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 4 & -3 & 4 & 8 \\ -6 & 1 & 1.5 & -4.5 \end{array}\right)$$

$$\xrightarrow{-2r_1+r_2} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ \boxed{2} & -1 & 2 & 2 \\ -6 & 1 & 1.5 & -4.5 \end{array}\right)$$

$$\xrightarrow{3r_1+r_3} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ \boxed{2} & -1 & 2 & 2 \\ \boxed{-3} & -2 & 4.5 & 4.5 \end{array}\right)$$

$$\xrightarrow{-2r_2+r_3} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ \boxed{2} & -1 & 2 & 2 \\ \boxed{-3} & \boxed{2} & .5 & .5 \end{array}\right)$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{(bottom up)}$$

(b) From last step in Gaussian elimination above, we get

$$A = LU = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ -3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ & -1 & 2 \\ & & .5 \end{pmatrix}$$

(c) When solving $Ax = (LU)x = b$, we need to do two steps

$$Ly = b$$

$$\begin{pmatrix} 1 & & \\ 2 & 1 & \\ -3 & 2 & 1 \end{pmatrix} y = \begin{pmatrix} 3 \\ 8 \\ -4.5 \end{pmatrix}$$

$$y = \begin{pmatrix} 3 \\ 2 \\ .5 \end{pmatrix} \quad \text{(top down)}$$

$$Ux = y$$

$$\begin{pmatrix} 2 & -1 & 1 \\ & -1 & 2 \\ & & .5 \end{pmatrix} x = \begin{pmatrix} 3 \\ 2 \\ .5 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{(bottom up)}$$

(d) GE with partial pivoting (must switch rows exactly):

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 4 & -3 & 4 & 8 \\ -6 & 1 & 1.5 & -4.5 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} -6 & 1 & 1.5 & -4.5 \\ 4 & -3 & 4 & 8 \\ 2 & -1 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{(2/3)r_1 + r_2} \left(\begin{array}{ccc|c} -6 & 1 & 1.5 & -4.5 \\ & -7/3 & 5 & 5 \\ 2 & -1 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{(1/3)r_1 + r_3} \left(\begin{array}{ccc|c} -6 & 1 & 1.5 & -4.5 \\ & -7/3 & 5 & 5 \\ & -2/3 & 3/2 & 3/2 \end{array}\right)$$

$$\xrightarrow{(-2/7)r_2 + r_3} \left(\begin{array}{ccc|c} -6 & 1 & 1.5 & -4.5 \\ & -7/3 & 5 & 5 \\ & & 1/14 & 1/14 \end{array}\right)$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{(bottom up)}$$

11. Find a nonzero vector x so that $\|Ax\|_\infty = \|A\|_\infty \|x\|_\infty$

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 3 & 1 & -3 \\ 2 & 1 & 0 \end{pmatrix}$$

Can we find a nonzero x so that $\|Ax\|_\infty = 2\|A\|_\infty \|x\|_\infty$?

- **<u>ans:</u>**

$$\|A\|_\infty = \max\{1+3+2, 3+1+3, 2+1+0\} = 7$$

Since the max is from row 2, we use the signs to construct an x of norm 1. Let

$$x = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$Ax = \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix}$$

$$\|Ax\|_\infty = \max\{4, 7, 3\} = 7$$

$$\|A\|_\infty \|x\|_\infty = (7)(1) = 7$$

No. We can never do it (unless $x = 0$), because

$$\|Ax\|_\infty \le \|A\|_\infty \|x\|_\infty$$

12. Find the relative forward and backward errors, and error magnification factors. Verify that the factors are no bigger than the condition number.

$$(A|b) = \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 2 & -3 & 5 \end{array}\right)$$

$$(a)\ x_c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad (b)\ x_c = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

- **<u>ans:</u>**

Easy to find the exact solution

$$x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(a)

$$x_c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f.r.e = \frac{\|x - x_c\|}{\|x\|} = \frac{2}{1} = 2$$

$$b.r.e = \frac{\|b - Ax_c\|}{\|b\|} = \frac{6}{5} = 1.2$$

error magnification factor:

$$\beta = \frac{\text{relative forward error}}{\text{relative backward error}} = 1.667$$

(b)

$$f.r.e = \frac{\|x - x_c\|}{\|x\|} = \frac{1}{1} = 1$$

$$b.r.e = \frac{\|b - Ax_c\|}{\|b\|} = \frac{2}{5} = 0.4$$

error magnification factor:

$$\beta = \frac{\text{relative forward error}}{\text{relative backward error}} = 2.5$$

$$\|A\|_\infty = 5$$

$$A^{-1} = \begin{pmatrix} 3/7 & 2/7 \\ 2/7 & -1/7 \end{pmatrix}$$

$$\|A^{-1}\|_\infty = 5/7$$

$$\text{Cond}(A) = \|A\|_\infty \|A^{-1}\|_\infty = 25/7 = 3.57$$

Yes. The condition is always no less than the error magnification factors.

13. Solve the following system $(A|b)$

$$\left(\begin{array}{ccc|c} 1 & -3 & 3 & 4 \\ -2 & 0 & 1 & -1 \\ 3 & -1 & -1 & 2 \end{array}\right)$$

(a) by GE without pivoting,

(b) by finding $A = LU$ and using it

(c) by GE with partial pivoting,

(d) by finding $PA = LU$ and using it

- **ans:**

(a) by GE without pivoting,

$$\begin{pmatrix} 1 & -3 & 3 & | & 4 \\ -2 & 0 & 1 & | & -1 \\ 3 & -1 & -1 & | & 2 \end{pmatrix}$$

$$\xrightarrow{2r_1+r_2} \begin{pmatrix} 1 & -3 & 3 & | & 4 \\ \boxed{-2} & -6 & 7 & | & 7 \\ 3 & -1 & -1 & | & 2 \end{pmatrix}$$

$$\xrightarrow{(-3)r_1+r_3} \begin{pmatrix} 1 & -3 & 3 & | & 4 \\ \boxed{-2} & -6 & 7 & | & 7 \\ \boxed{3} & 8 & -10 & -10 & \end{pmatrix}$$

$$\xrightarrow{(4/3)r_2+r_3} \begin{pmatrix} 1 & -3 & 3 & | & 4 \\ \boxed{-2} & -6 & 7 & | & 7 \\ \boxed{3} & \boxed{-4/3} & -2/3 & -2/3 & \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{(bottom up)}$$

(b) From last step in Gaussian elimination above, we get

$$A = LU = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ 3 & -4/3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 \\ & -6 & 7 \\ & & -2/3 \end{pmatrix}$$

(please check it.)

When solving $Ax = (LU)x = b$, we need to do two steps

$$Ly = b$$

$$\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 3 & -4/3 & 1 \end{pmatrix} y == \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

$$y = \begin{pmatrix} 4 \\ 7 \\ -2/3 \end{pmatrix} \quad \text{(top down)}$$

$$Ux = y$$

$$\begin{pmatrix} 1 & -3 & 3 \\ & -6 & 7 \\ & & -2/3 \end{pmatrix} x = \begin{pmatrix} 4 \\ 7 \\ -2/3 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{(bottom up)}$$

(c) GE with partial pivoting (must switch rows exactly):

$$\begin{pmatrix} 1 & -3 & 3 & | & 4 \\ -2 & 0 & 1 & | & -1 \\ 3 & -1 & -1 & | & 2 \end{pmatrix}$$

$$\xrightarrow{r_1\leftrightarrow r_3} \begin{pmatrix} 3 & -1 & -1 & | & 2 & r_3 \\ -2 & 0 & 1 & | & -1 & r_2 \\ 1 & -3 & 3 & | & 4 & r_1 \end{pmatrix}$$

$$\xrightarrow{(2/3)r_1+r_2} \begin{pmatrix} 3 & -1 & -1 & | & 2 & r_3 \\ \boxed{-2/3} & -2/3 & 1/3 & | & 1/3 & r_2 \\ 1 & -3 & 3 & | & 4 & r_1 \end{pmatrix}$$

$$\xrightarrow{(-1/3)r_1+r_3} \begin{pmatrix} 3 & -1 & -1 & | & 2 & r_3 \\ \boxed{-2/3} & -2/3 & 1/3 & | & 1/3 & r_2 \\ \boxed{1/3} & -8/3 & 10/3 & | & 10/3 & r_1 \end{pmatrix}$$

$$\xrightarrow{r_2\leftrightarrow r_3} \begin{pmatrix} 3 & -1 & -1 & | & 2 & r_3 \\ \boxed{1/3} & -8/3 & 10/3 & | & 10/3 & r_1 \\ \boxed{-2/3} & -2/3 & 1/3 & | & 1/3 & r_2 \end{pmatrix}$$

$$\xrightarrow{(-1/4)r_2+r_3} \begin{pmatrix} 3 & -1 & -1 & | & 2 & r_3 \\ \boxed{1/3} & -8/3 & 10/3 & | & 10/3 & r_1 \\ \boxed{-2/3} & \boxed{1/4} & -1/2 & | & -1/2 & r_2 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{(bottom up)}$$

(d) by finding $PA = LU$ and using it
From above work, we have

$$P = \begin{pmatrix} 0 & 0 & 1 & r_3 \\ 1 & 0 & 0 & r_1 \\ 0 & 1 & 0 & r_2 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & \\ 1/3 & 1 & \\ -2/3 & 1/41 & \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & -1 & -1 \\ & -8/3 & 10/3 \\ & & -1/2 \end{pmatrix}$$

$$PA = LU$$

(check it.) Now, we have three steps:

$$z = Pb \qquad \Rightarrow z = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

$$Ly = z \qquad \Rightarrow y = \begin{pmatrix} 2 \\ 10/3 \\ -1/2 \end{pmatrix}$$

$$Ux = y \qquad \Rightarrow x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

14. Solve the following system $(A|b)$

$$\begin{pmatrix} 2 & 1 & -4 & | & -7 \\ 1 & -1 & 1 & | & -2 \\ -1 & 3 & -2 & | & 6 \end{pmatrix}$$

(a) by GE without pivoting,
(b) by finding $A = LU$ and using it
(c) by GE with partial pivoting, (row pivoting every step)
(d) by finding $PA = LU$ and using it

- **ans:** (a) All row operations must be exactly as shown here.

$$(A|b) \to (U|c)$$

$(-1/2)r_1 + r_2 \to r_2$, $(1/2)r_1 + r_3 \to r_3$:

$$\begin{pmatrix} 2 & 1 & -4 & | & -7 \\ \boxed{1/2} & -3/2 & 3 & | & 3/2 \\ \boxed{-1/2} & 7/2 & -4 & | & 5/2 \end{pmatrix}$$

$(7/3)r_2 + r_3 \to r_3$:

$$\begin{pmatrix} 2 & 1 & -4 & | & -7 \\ \boxed{1/2} & -3/2 & 3 & | & 3/2 \\ \boxed{-1/2} & \boxed{-7/3} & 3 & | & 6 \end{pmatrix}$$

Backward substitution, starting from equation 3:

$$x = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

(b) By (a)

$$L = \begin{pmatrix} 1 & & \\ 1/2 & 1 & \\ -1/2 & -7/3 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 1 & -4 \\ & -3/2 & 3 \\ & & 3 \end{pmatrix}$$

$$A = LU$$

Solve the system in two steps:

$$Ly = b \qquad \Rightarrow y = \begin{pmatrix} -7 \\ 3/2 \\ 6 \end{pmatrix}$$

$$Ux = y \qquad \Rightarrow x = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

(c) GE with partial pivoting (must switch rows exactly):

$$(A|b) \to (U|c)$$

$(-1/2)r_1 + r_2 \to r_2$, $(1/2)r_1 + r_3 \to r_3$:

$$\begin{pmatrix} 2 & 1 & -4 & | & -7 & r_1 \\ \boxed{1/2} & -3/2 & 3 & | & 3/2 & r_2 \\ \boxed{-1/2} & 7/2 & -4 & | & 5/2 & r_3 \end{pmatrix}$$

$r_2 \leftrightarrow r_3$:

$$\begin{pmatrix} 2 & 1 & -4 & | & -7 & r_1 \\ \boxed{-1/2} & 7/2 & -4 & | & 5/2 & r_3 \\ \boxed{1/2} & -3/2 & 3 & | & 3/2 & r_2 \end{pmatrix}$$

$(3/7)r_2 + r_3$:

$$\begin{pmatrix} 2 & 1 & -4 & | & -7 & r_1 \\ \boxed{-1/2} & 7/2 & -4 & | & 5/2 & r_3 \\ \boxed{1/2} & \boxed{-3/7} & 9/7 & | & 18/7 & r_2 \end{pmatrix}$$

$$x = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \quad \text{(bottom up)}$$

(d) By finding $PA = LU$ and using it

From above work, we have

$$P = \begin{pmatrix} 1 & 0 & 0 & r_1 \\ 0 & 0 & 1 & r_3 \\ 0 & 1 & 0 & r_2 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & \\ -1/2 & 1 & \\ 1/2 & -3/7 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 1 & -4 \\ & 7/2 & -4 \\ & & 9/7 \end{pmatrix}$$

$$PA = LU$$

(check it.) Now, we have three steps:

$$z = Pb \qquad \Rightarrow z = \begin{pmatrix} -7 \\ 6 \\ -2 \end{pmatrix}$$

$$Ly = z \qquad \Rightarrow y = \begin{pmatrix} -7 \\ 5/2 \\ 18/7 \end{pmatrix}$$

$$Ux = y \qquad \Rightarrow x = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$