M353 7.1-2 (S. Zhang) (nonlinear shooting).

1. Solve the boundary value problem (finding $x(1)$):

$$x'' - x' = 2e^x, \qquad x(0) = 2, \quad x(2) = 20$$

(a) by the Euler **nonlinear** shooting method with $h = 1$ and 2 steps (5 shootings) of the bisection method on $x'(0) \in [0, 2]$.

(b) by the finite difference method with $h = 1$; with 4 steps of the Newton's method starting with $x_1 = 4$.

- **ans:**

(a) Convert the second order equation to a first order system:

$$\begin{aligned} x' &= y \\ y' &= x'' = y + 2e^x \end{aligned}$$

$$Z = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$Z' = F(t, Z) = \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} y \\ y + 2e^x \end{pmatrix}$$

Discretization:

$$h = 1, \; t_0 = 0, \; t_1 = 1, \; t_2 = 2$$

The key job is to find $x'(0)$. So we start with two tries $x'(0) = 0$ and $x'(0) = 2$.

(Note that, for linear shooting problems, two tries are enough to find the correct shooting. But for nonlinear shooting problems, we have to try many many times, each time, we try again shooting in the middle — bisection method.)

Shooting 1: $x' = 0$ (the lower end point)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 14.7781 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 14.7781 \end{pmatrix} \\ t_1 &= t_0 + h = 1 \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= \begin{pmatrix} 16.7781 \\ 44.3343 \end{pmatrix} \end{aligned}$$

$x(2) \sim x_2 = Z_{2,1} = 16.7781$. We shoot too low (target $x(2) = 20$).

Shooting 2: $x' = 2$ (the high end point)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 16.7781 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 18.7781 \end{pmatrix} \\ t_1 &= t_0 + h = 1 \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= \begin{pmatrix} 22.7781 \\ 146.7525 \end{pmatrix} \end{aligned}$$

$x(2) \sim x_2 = Z_{2,1} = 22.7781$. We shoot too high (target $x(2) = 20$).

So, we know we need to choose a shooting angle $x'(0)$ between 0 and 2. We apply bisection method.

Shooting 3: $x' = \frac{0+2}{2} = 1$ (the middle point)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1.0000 \\ 15.7781 \end{pmatrix} \\ &= \begin{pmatrix} 3.0000 \\ 16.7781 \end{pmatrix} \\ t_1 &= t_0 + h = 1 \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= \begin{pmatrix} 19.7781 \\ 73.7273 \end{pmatrix} \end{aligned}$$

$x(2) \sim x_2 = Z_{2,1} = 19.7781$. We shoot too low (target $x(2) = 20$).

What do we try next? Between $(0, 1)$ or between $(1, 2)$?

Since the target is between the shooting $x'(0) = 1$ and the shooting $x'(0) = 2$, we use $(1, 2)$ for the next shooting.

Shooting 4: $x' = \frac{1+2}{2} = 1.5$ (the middle point of $(1, 2)$)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 2 \\ 1.5 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 2 \\ 1.5 \end{pmatrix} + \begin{pmatrix} 1.5000 \\ 16.2781 \end{pmatrix} \\ &= \begin{pmatrix} 3.5000 \\ 17.7781 \end{pmatrix} \\ t_1 &= t_0 + h = 1 \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= \begin{pmatrix} 21.2781 \\ 101.7871 \end{pmatrix} \end{aligned}$$

We shoot too high.

We finished two bisection iterations. We know the right shooting $x'(0)$ would be between 1 and 1.5. To find the $x(1)$ for the best shooting, we compute one more time.

Shooting 5: $x' = \frac{1+1.5}{2} = 1.25$ (the middle point of $(1, 1.5)$)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 2 \\ 1.25 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 2 \\ 1.25 \end{pmatrix} + \begin{pmatrix} 1.25000 \\ 16.0281 \end{pmatrix} \\ &= \begin{pmatrix} 3.2500 \\ 17.2781 \end{pmatrix} \\ t_1 &= t_0 + h = 1 \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= \begin{pmatrix} 20.5281 \\ 86.1369 \end{pmatrix} \end{aligned}$$

We shoot a little too high, but close to the target 20, than all other shootings.

So we find only one approximate value

$$x(1) \sim Z_{1,1} = 3.25$$

(b) Finite difference method:

Discretization:

$$h = 1,\ t_0 = 0,\ t_1 = 1,\ t_2 = 2$$

We look for only one unknown $x_1 \simeq x(1)$. At $t = t_1 = 1$: we replace the equation by the finite differences:

$$\begin{aligned} x'' - x' &= 2e^x, \quad x(0) = 2, \quad x(2) = 20 \\ \frac{x_0 - 2x_1 + x_2}{h^2} - \frac{-x_0 + x_2}{2h} &= 2e^{x_1}, \qquad x_0 = 2, \qquad x_2 = 20 \end{aligned}$$

$$\begin{aligned} \frac{2 - 2x_1 + 20}{1^2} - \frac{-2 + 20}{2} &= 2e^{x_1} \\ -2x_1 + 22 - 9 &= 2e^{x_1} \\ 13 - 2x_1 - 2e^{x_1} &= 0 \end{aligned}$$

We define a function $f(z)$ and use Newton's method to find the root of $f(z)$:

$$\begin{aligned} f(z) &= 13 - 2z - 2e^z \\ f'(z) &= -2 - 2e^z \\ z_0 &= 4 \\ z_1 &= z_0 - \frac{f(z_0)}{f'(z_0)} = 3.0630 \\ z_2 &= z_1 - \frac{f(z_1)}{f'(z_1)} = 2.2611 \\ z_3 &= 1.7556 \\ z_4 &= 1.6020 \\ z_5 &= 1.5911 \\ z_6 &= 1.5911 \end{aligned}$$

(we can stop at z_4. The rest is for checking.)

So the approximation solution for $x(1)$ is

$$x(1) \sim z_6 = 1.5911$$

Note that, because the step size is way too big, the two methods give two huge different solutions.

But recall that Finite difference method is of second order, but the Euler shooting method is of first order (need to know them for exatrapolations.)