M353 Hw 12 (S. Zhang) 8.2 .

1. 8.2:a1,a2

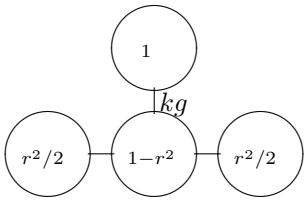
84.99
1. (8.2:a1) Solve the wave equation by the finite difference
84.24 method with $h = 1/3$ and $k = 0.1$.

$$u_{tt} = 4u_{xx};\ t \in (0, 0.2),\ x \in (0, 1)$$

with initial and boundary values

$$u(x, 0) = 1, \quad u_t(x, 0) = 2x,$$
$$u(0, t) = 0, \quad u(1, t) = 0.$$

• **ans:**



$$r = c\frac{k}{h} = c\frac{\Delta t}{\Delta x} = .6.$$

$$A = \text{triDiag}\begin{pmatrix} r^2 & 2-2r^2 & r^2 \end{pmatrix}$$

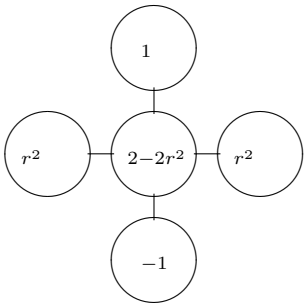
$$A = \begin{pmatrix} 1.28 & .36 \\ .36 & 1.28 \end{pmatrix}$$

$$x = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$U_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 2x \end{pmatrix} = \begin{pmatrix} 2/3 \\ 4/3 \end{pmatrix}$$

$$U_1 = \frac{1}{2}AU_0 + kG$$
$$= \begin{pmatrix} .82 \\ .82 \end{pmatrix} + \begin{pmatrix} .0667 \\ .1333 \end{pmatrix} = \begin{pmatrix} .8867 \\ 0.9533 \end{pmatrix}$$



<u>High levels</u>:

$$U_2 = 2AU_1 - U_0$$
$$= \begin{pmatrix} 0.8062 \\ 0.8626 \end{pmatrix} - \begin{pmatrix} .8867 \\ 0.9533 \end{pmatrix} = \begin{pmatrix} .4781 \\ 0.5395 \end{pmatrix}$$

2. (8.2:a2) Solving the wave equation by the finite differ-
84.4 ence method.

$$u_{tt} = 4u_{xx}, \quad t \in (0, .4),\ x \in (0, 1)$$

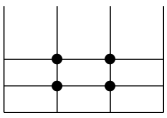
with initial and boundary values

$$u(x, 0) = \sin(\pi x) + \sin(2\pi x),$$
$$u_t(x, 0) = x,$$
$$u(0, t) = 0,$$
$$u(1, t) = 0.$$

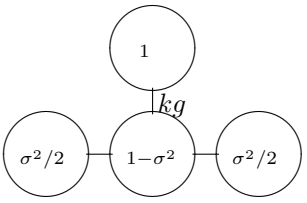
(a) Let $h = \Delta x = \frac{1}{3}$, $k = \Delta t = 0.1$. Find $u(x_i, 0.4)$.
(b) Let $h = \Delta x = \frac{1}{4}$, $k = \Delta t = 0.1$. Find $u(x_i, 0.2)$.

• **ans:**

(a) Grids:



$$\sigma = c\frac{k}{h} = c\frac{\Delta t}{\Delta x} = .6.$$



$$x_i = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

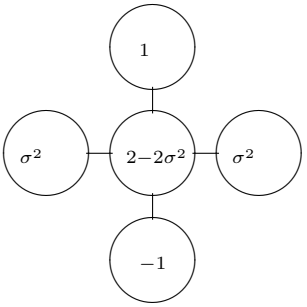
$$U_0 = \begin{pmatrix} \sin(\pi x) + \sin(2\pi x) \end{pmatrix} = \begin{pmatrix} 1.732 \\ 0 \end{pmatrix}$$

$$G = \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$A = \begin{pmatrix} 2-2\sigma^2 & -\sigma^2 \\ \sigma^2 & 2-2\sigma^2 \end{pmatrix} = \begin{pmatrix} 1.28 & .36 \\ 0.36 & 1.28 \end{pmatrix}$$

$$U_1 = \frac{1}{2}AU_0 + kG = \begin{pmatrix} 1.14184 \\ 0.37843 \end{pmatrix}$$

<u>High levels</u>:



$$U_2 = AU_1 - U_0 = \begin{pmatrix} -0.13425 \\ 0.8954 \end{pmatrix}$$

$$U_3 = AU_2 - U_1 = \begin{pmatrix} -0.9913 \\ 0.71942 \end{pmatrix}$$

$$U_4 = AU_3 - U_2 = \begin{pmatrix} -0.8756 \\ -0.33147 \end{pmatrix}$$

Draw a graph to show the wave by U_i.

(b)

$$\sigma = c\frac{k}{h} = c\frac{\Delta t}{\Delta x} = .8.$$

$$x = \begin{pmatrix} 1/4 \\ 1/2 \\ 3/4 \end{pmatrix}$$

$$U_0 = \big(\sin(\pi x) + \sin(2\pi x)\big) = \begin{pmatrix} 1.7071 \\ 1 \\ -0.2929 \end{pmatrix}$$

$$G = \big(x\big) = \begin{pmatrix} 1/4 \\ 1/2 \\ 3/4 \end{pmatrix}$$

$$\begin{aligned} A &= tridiag\begin{pmatrix} \sigma^2 & 2-2\sigma^2 & \sigma^2 \end{pmatrix} \\ &= \begin{pmatrix} 0.72 & .64 & \\ .64 & 0.72 & .64 \\ & .64 & 0.72 \end{pmatrix} \end{aligned}$$

$$U_1 = \frac{1}{2}AU_0 + kG = \begin{pmatrix} 0.9596 \\ 0.8625 \\ 0.2896 \end{pmatrix}$$

$$U_2 = AU_1 - U_0 = \begin{pmatrix} -0.4642 \\ 0.4205 \\ 1.0534 \end{pmatrix}$$

$(U_2)_i$ approximates $u(x_i, 0.2)$.
