M353 Hw 11 (S. Zhang) 8.1.

1. (8.1:a1) Solving the heat equation by the finite differ-
82.28 ence method. $u_t = 4u_{xx}$ for $t \in (0, 0.1)$ and $x \in (0, 3)$ with initial and boundary values

$$u(x, 0) = 2(3 - x)x, \quad u(0, t) = 0, \quad u(3, t) = 0.$$

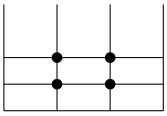
Let $\Delta t = 0.1$. Find $u(x_i, 0.1)$ by the following methods with $h = 1$.

(a) the Euler (explicit) method.

(b) the backward Euler (implicit) method.

(c) the Crank-Nicholson method.

- **ans:**

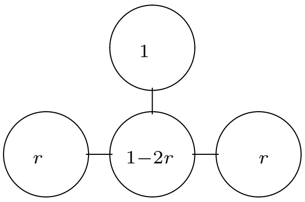
We cut the domain this way : 

For all three methods

$$r = c^2 \frac{k}{h^2} = c^2 \frac{\Delta t}{\Delta x^2} = 0.4$$

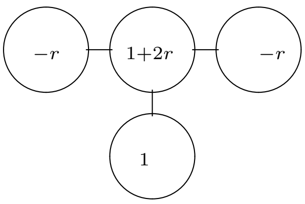
$$U_0 = \begin{pmatrix} f(x_1) \\ f(x_2) \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

Euler method:



$$U_1 = AU_0 = \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.2 \end{pmatrix} U_0 = \begin{pmatrix} 2.4 \\ 2.4 \end{pmatrix}$$

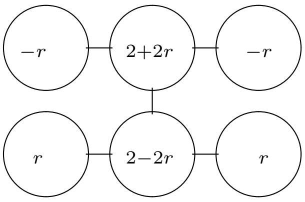
Backward Euler:



$$AU_1 = U_0, \; A = \begin{pmatrix} 1.8 & -0.4 \\ -0.4 & 1.8 \end{pmatrix}$$

$$\begin{aligned} U_1 &= A^{-1}U_0 \\ &= \begin{pmatrix} 0.58442 & 0.12987 \\ 0.12987 & 0.58442 \end{pmatrix} U_0 \\ &= \begin{pmatrix} 2.8571 \\ 2.8571 \end{pmatrix} \end{aligned}$$

Crank-Nicholson:



$$\begin{aligned} AU_1 &= BU_0 \\ A &= \begin{pmatrix} 2.8 & -0.4 \\ -0.4 & 2.8 \end{pmatrix}, \text{(see the graph)} \\ B &= \begin{pmatrix} 1.2 & 0.4 \\ 0.4 & 1.2 \end{pmatrix}, \\ A^{-1}B &= \begin{pmatrix} 0.45833 & 0.20833 \\ 0.20833 & 0.45833 \end{pmatrix} \\ U_1 = A^{-1}BU_0 &= \begin{pmatrix} 2.6667 \\ 2.6667 \end{pmatrix} \end{aligned}$$

2. (8.1:a2) Solving the heat equation by the finite differ-
82.6 ence method. $u_t = 4u_{xx}$ for $t \in (0, 0.1)$ and $x \in (0, 1)$ with initial and boundary values

$$\begin{aligned} u(0, t) &= 0, \quad u(1, t) = 0.\text{(initial conditions)} \\ u(x, 0) &= 2x, \quad \text{(BC, use } x(1, 0) = 0 \text{ from the IC above)} \end{aligned}$$

Let $k = \Delta t = 0.1$. Find $u(x_i, 0.1)$ by the following methods with $h = 1/2$ and with $h = 1/3$ again.

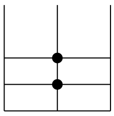
(a) the Euler (explicit) method .

(b) the backward Euler (implicit) method.

(c) the Crank-Nicholson method.

- **ans:** Euler method $h = 1/2$:

We cut the domain this way:



We replace the differential equation at point (x_i, t_j) by

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

or

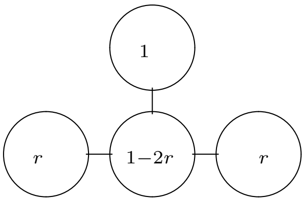
$$u_{i,j+1} = ru_{i-1,j} + (1 - 2r)u_{i,j} + ru_{i+1,j}$$

where

$$r = c^2 \frac{k}{h^2} = c^2 \frac{\Delta t}{\Delta x^2} = 1.6$$

So matrix A is tridiagonal shape like:

$$A = \begin{pmatrix} r & 1 - 2r & r \end{pmatrix}$$



$$U_0 = \begin{pmatrix} 0 \\ 2x \\ 0 \end{pmatrix}$$

Since we will always have 0 boundary condition, we omit the two compoents at the ends.

$$U_0 = \begin{pmatrix} 2x \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -2.2 \end{pmatrix}$$

$$U_1 = AU_0 = \left(\begin{pmatrix} -2.2 \end{pmatrix}\right) \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} -2.2 \end{pmatrix}$$

Bad, unstasble solution.

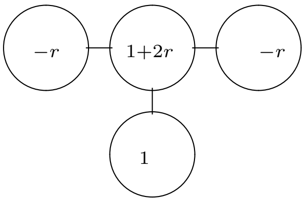
<u>Backward Euler</u>:

We replace the differential equation at point (x_i, t_j) by

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c^2 \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}$$

or

$$-ru_{i-1,j+1} + (1 + 2r)u_{i,j+1} - ru_{i+1,j+1} = u_{i,j}$$



$$AU_1 = U_0, \ A = \begin{pmatrix} 4.2 \end{pmatrix}$$

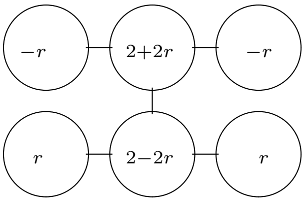
$$\begin{aligned} U_1 &= A^{-1}U_0 \\ &= \begin{pmatrix} 0.2381 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \qquad = \begin{pmatrix} 0.2381 \end{pmatrix} \end{aligned}$$

Good, stasble solution.

<u>Crank-Nicholson</u>:

We replace the differential equation at point (x_i, t_j) by

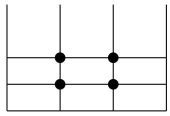
$$\begin{aligned} \frac{u_{i,j+1} - u_{i,j}}{k} =& c^2 \frac{1}{2} \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \\ &+ c^2 \frac{1}{2} \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} \end{aligned}$$



$$\begin{aligned} AU_1 &= BU_0 \\ A &= \begin{pmatrix} 5.2 \end{pmatrix}, \\ B &= \begin{pmatrix} -1.2 \end{pmatrix}, \\ A^{-1}B &= \begin{pmatrix} -0.23077 \end{pmatrix} \\ U_1 &= A^{-1}BU_0 = \begin{pmatrix} -0.2308 \end{pmatrix} \end{aligned}$$

Bad, unstasble solution.

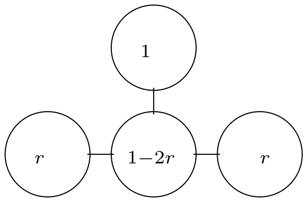
<u>Euler method $h = 1/3$</u>:

We cut the domain this way: 

<u>Euler method</u>: For all three methods

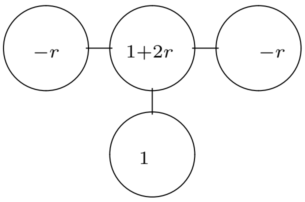
$$r = c^2 \frac{k}{h^2} = c^2 \frac{\Delta t}{\Delta x^2} = 3.6$$

<u>For Euler</u> :



$$\begin{aligned} U_1 = AU_0 &= \left(\begin{pmatrix} -6.2 & 3.6 \\ 3.6 & -6.2 \end{pmatrix}\right) \begin{pmatrix} 2/3 \\ 4/3 \end{pmatrix} \\ &= \begin{pmatrix} .6667 \\ -5.8667 \end{pmatrix} \end{aligned}$$

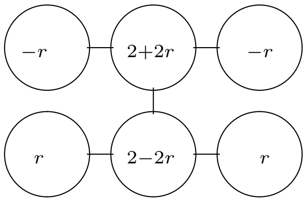
<u>Backward Euler</u>:



$$AU_1 = U_0, \ A = \begin{pmatrix} 8.2 & -3.6 \\ -3.6 & 8.2 \end{pmatrix}$$

$$\begin{aligned} U_1 &= A^{-1}U_0 \\ &= \begin{pmatrix} 0.15107 & 0.066323 \\ 0.066323 & 0.15107 \end{pmatrix} \begin{pmatrix} 2/3 \\ 4/3 \end{pmatrix} \\ &= \begin{pmatrix} .1891 \\ .2456 \end{pmatrix} \end{aligned}$$

Crank-Nicholson:



$$
\begin{aligned}
AU_1 &= BU_0 \\
A &= \begin{pmatrix} 9.2 & -3.6 \\ -3.6 & 9.2 \end{pmatrix}, \\
B &= \begin{pmatrix} -5.2 & 3.6 \\ 3.6 & -5.2 \end{pmatrix}, \\
A^{-1}B &= \begin{pmatrix} -0.48661 & 0.20089 \\ 0.20089 & -0.48661 \end{pmatrix} \\
U_1 = A^{-1}BU_0 &= \begin{pmatrix} -0.0565 \\ -0.5149 \end{pmatrix}
\end{aligned}
$$

3. (8.1:a3) Solving the heat equation by the finite differ-
82.29 ence method. $u_t = u_{xx}$ for $t \in (0, 0.2)$ and $x \in (1, 5)$ with initial and boundary values

$$u(x,0) = (5-x)(1-x), \quad u(1,t) = 0, \quad u(5,t) = 0.$$

(a) the Euler (explicit) method. $h = 1$ and $k = 0.1$.

(b) the backward Euler (implicit) method. $h = 1$ and $k = 0.1$.