M353 Hw 10) (S. Zhang) 6.6, 7.1-2.

1. 6.6: 1bc, 2, a1

71.99
1. (6.6:1c) Find the exact solution. Apply backward Euler
71.12 with $h = 1/4$ for $y(1)$, find the error.

$$y' = 4t - 2y, \ y(0) = 0$$

- **ans:**

1st order linear: $y' + py = q$.

$$\mu = e^{\int p} = e^{2t}$$

sol:

$$y = \mu^{-1} \int \mu q = e^{-2t} \int e^{2t} 4t dt$$
$$= e^{-2t}(e^{2t}2t - e^{2t} + C) = 2t - 1 + Ce^{-2t}$$

$$y(0) = 0 \ \Rightarrow \ y = 2t - 1 + e^{-2t}$$

$$y(1) = 1 + e^{-2} = 1.1353$$

Backward Euler: $y' = f(t, y)$.

$$y_1 = y_0 + hf_1 = y_0 + hf(t_1, y_1)$$
$$= y_0 + 4ht_1 - 2hy_1$$

$$y_1 = \frac{y_0 + 4ht_1}{1 + 2h}$$

$$y_{i+1} = \frac{y_i + 4ht_{i+1}}{1 + 2h}$$

$$h = 1/4, ; \ y_0 = 0, \ t_0 = 0$$

$$t_1 = 0.25; \quad y_1 = \frac{0 + 4(0.25)(0.25)}{1.5}$$
$$= 0.1667$$
$$err = y(t_i) - y_i = -0.0601$$

$$i = 2; \ t_i = 0.5$$
$$y_i = 0.4444$$
$$err = y(t_i) - y_i = -0.0766$$

$$i = 3; \ t_i = 0.75$$
$$y_i = 0.7963$$
$$err = y(t_i) - y_i = -0.0732$$

$$i = 4; \ t_i = 1$$
$$y_i = 1.1975$$
$$err = y(t_i) - y_i = -0.0622$$

2. (6.6:2c) Find the equilibrium solutions and the value of
71.15 the Jacobian at the equilibria. Is the equation stiff?

$$y' = -10 \sin y$$

- **ans:** Let $y' = 0$. By the DE,

$$0 = -10 \sin y, \ y = 0, k\pi$$

The Jacobian:

$$f_y(t, y) = \frac{d}{dy}(-10 \sin y) = -10 \cos y$$

At $y = 0$

$$f_y(t, y) = -10$$

We say the equation is stiff. (If the Jacobian value is less than -10 or so, we say the equation is stiff).

At $y = \pi$

$$f_y(t, y) = 10$$

We say the equation is not stiff. (If the Jacobian value is less than -10 or so, we say the equation is stiff).

3. (6.6:a1)
71.16

$$y' = 1 - 20y, \ 0 \le t \le 0.2; \ y(0) = 1/2$$

(1) Solve the IVP,
(2) Apply Euler $h = 0.1$, find error.
(3) Apply backward Euler with $h = 0.1$, find error.
(4) Apply backward Euler with $h = 0.2$, find error.
(4) Apply extrapolation on the backward Euler data, find error.

- **ans:** 1st order linear:

$$y' + 20y = 1$$

$$\mu = e^{\int p} = e^{20t}$$

sol:

$$y = \mu^{-1} \int \mu q = e^{-20t} \int e^{20t} dt$$
$$= e^{-20t}(\frac{1}{20}e^{20t} + C) = \frac{1}{20} + Ce^{-20t}$$

By $y(0) = 1/2$

$$y = \frac{1}{20} + \frac{9}{20}e^{-20t}, \ y(0.2) = 0.0418$$

Euler with $h = 0.1$:

$$y_{i+1} = y_i + hf_i = y_i + hf(t_i, y_i)$$

$$y_1 = y_0 + h(1 - 20y_0)$$

$$
\begin{aligned}
h = .1,\ y_0 &= 1/2 \\
y_1 &= 1/2 + 0.1(1-10) = -0.4, \\
err &= y(0.1) - y_1 \\
&= 0.3891
\end{aligned}
$$

$$
\begin{aligned}
t_1 = .1,\ y_2 &= 0.5, \\
err &= y(0.2) - y_2 \\
&= -0.4582
\end{aligned}
$$

Backward Euler with $h = 0.1$:

$$y_1 = y_0 + hf_1 = y_0 + hf(t_1, y_1)$$

$$y_1 = y_0 + h - 20hy_1$$

$$y_1 = \frac{y_0 + h}{1 + 20h}$$

$$
\begin{aligned}
h = .1, y_0 &= 1/2 \\
y_1 &= \frac{0.5 + 0.1}{3} = 0.2, \\
err &= y(0.1) - y_1 \\
&= -0.2109
\end{aligned}
$$

$$
\begin{aligned}
t_1 = .1,\ y_2 &= \frac{y_1 + h}{1 + 20h} = 0.1, \\
err &= y(0.2) - y_2 \\
&= -0.0582
\end{aligned}
$$

The Backward Euler method is better.
Backward Euler with $h = 0.2$:

$$y_1 = y_0 + hf_1 = y_0 + hf(t_1, y_1)$$

$$y_1 = y_0 + h - 20hy_1$$

$$y_1 = \frac{y_0 + h}{1 + 20h}$$

$$
\begin{aligned}
h = .2, y_0 &= 1/2 \\
y_1 &= 0.14, \\
err &= y(0.2) - y_1 \\
&= -0.0982
\end{aligned}
$$

Extrapolation on backward Euler data:

$$
\begin{aligned}
y_{extrap} &= \frac{2y_{2,h=0.1} - y_{1,h=0.2}}{2 - 1} = 0.06 \\
err &= -0.0182 \ \text{(Much better than last two.)}
\end{aligned}
$$

---

1. (7.1-2:a1) Solve the boundary value problem by the Eu-
72.15 ler shooting method with $h = 1/2$ (find approximate values $x(1/2)$) :

$$x'' - x' = 2 - 2t, \qquad x(0) = 2, \quad x(1) = 3$$

Then solve it by the finite difference method with $h = 1/2$, and $h = 1/3$. The exact solution is given:

$$x = t^2 + 2$$

- **ans:** Exact solution is

$$x(1/2) = 2.25$$

Shooting 1 – For u:

$$x'' - x' = 2 - 2t,\ y' = x'' = y + 2 - 2t$$

$h = 1$,

$$U = Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$U_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

$$U' = F(t, U) = \begin{pmatrix} y \\ y + 2 - 2t \end{pmatrix}$$

$$
\begin{aligned}
U_1 &= U_0 + hF(t_0, U_0) \\
&= U_0 + h\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= U_1 + hF(t_1, U_1) \\
&= U_1 + h\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2 \end{pmatrix}
\end{aligned}
$$

Shooting 2 – For v:

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x'' - x' = 0,\ y' = x'' = y$$

$$V' = F(t, V) = \begin{pmatrix} y \\ y \end{pmatrix}$$

$$
\begin{aligned}
V_1 &= V_0 + hF(t_0, V_0) \\
&= V_0 + h\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} .5 \\ 1.5 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
V_2 &= V_1 + hF(t_1, V_1) \\
&= V_1 + h\begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 2.25 \end{pmatrix}
\end{aligned}
$$

<u>Combine them:</u>

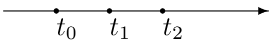
$$\begin{aligned} x(t) &= u + \frac{\beta - u(b)}{v(b)} v \\ &= u + \frac{3-2.5}{1.25} v \\ &= u + 0.4v \end{aligned}$$

In particular,

$$\begin{aligned} x(1/2) &\sim u(1/2) + 0.4v(1/2) = (U_1)_1 + 0.4(V_1)_1 \\ &= 2.2 \end{aligned}$$

Error is 0.05.

<u>Finite difference for $h = 1/2$:</u>

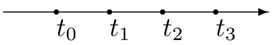


When $h = 1/2$, only 1 unknown x_1. x_0 and x_2 and known.

$$\begin{aligned} &\frac{x_0 - 2x_1 + x_2}{h^2} - \frac{x_2 - x_0}{2h} = 2 - 2t_1 \\ &(-2)x_1 = -4.5 \\ &x_1 = 2.25 \end{aligned}$$

The error is 0!

<u>Finite difference for $h = 1/3$:</u>



When $h = 1/3$, 2 unknowns x_1, x_2. x_0 and x_3 are the two given boundary values. The 2 equations are similar to the one above except h is different and all indices are increased by one.

$$\begin{aligned} &\frac{x_0 - 2x_1 + x_2}{h^2} - \frac{x_2 - x_0}{2h} = 2 - 2t_1 \\ &\frac{x_1 - 2x_2 + x_3}{h^2} - \frac{x_3 - x_1}{2h} = 2 - 2t_2 \end{aligned}$$

We would get the following linear system:

$$\begin{pmatrix} -2 & 0.8333 \\ 1.1667 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2.1852 \\ -2.4259 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2.111 \\ 2.4444 \end{pmatrix}$$

The errors are

$$x(1/3) - 2.111 = (\frac{1}{3})^2 + 2 - 2.111 = 0$$

$$x(2/3) - 2.444 = (\frac{2}{3})^2 + 2 - 2.444 = 0$$

2. (7.1-2:a2) Solve the boundary value problem by the Eu-
72.14 ler shooting method with $h = 1$ (find approximate values $x(1)$) :

$$x'' - x' - 2x = -2t - 5, \qquad x(0) = 2, \quad x(2) = 4$$

Then solve it by the finite difference method with $h = 1$, and $h = 1/2$. Extrapolate the solutions here.

- **ans:** <u>Shooting 1 – For u:</u> $h = 1$,

$$U = Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$U_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

$$U' = F(t, U) = \begin{pmatrix} y \\ y + 2 * x - 2 * t - 5 \end{pmatrix}$$

$$\begin{aligned} U_1 &= U_0 + hF(t_0, U_0) \\ &= U_0 + h \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} U_2 &= U_1 + hF(t_1, U_1) \\ &= U_1 + h \begin{pmatrix} -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \end{aligned}$$

<u>Shooting 2 – For v:</u>

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V' = F(t, V) = \begin{pmatrix} y \\ y + 2 * x \end{pmatrix}$$

$$\begin{aligned} V_1 &= V_0 + hF(t_0, V_0) \\ &= V_0 + h \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} V_2 &= V_1 + hF(t_1, V_1) \\ &= V_1 + h \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \end{aligned}$$

<u>Combine them:</u>

$$\begin{aligned} x(t) &= u + \frac{\beta - u(b)}{v(b)} v \\ &= u + \frac{4-1}{3} v \\ &= u + v \end{aligned}$$

In particular,

$$\begin{aligned} x(1/2) &\sim u(1/2) + v(1/2) = (U_1)_1 + (V_1)_1 \\ &= 3 \end{aligned}$$

Error is 0. When $h = 1$, only 1 unknown x_1. x_0 and x_2 and known.

$$\begin{aligned} &\frac{x_0 - 2x_1 + x_2}{h^2} - \frac{x_2 - x_0}{2h} - 2x_1 = -2t_1 - 5 \\ &(-14)x_1 = -12 \\ &x_1 = 3 \end{aligned}$$

When $h = 1/2$, 3 unknowns x_1, x_2, x_3. The 3 equations are similar to the one above except h is different and all indices are increased by one. We would get the following linear system:

$$\begin{pmatrix} -2.5 & 0.75 & \\ 1.25 & -2.5 & 0.75 \\ & 1.25 & -2.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -1.75 \\ -5 \end{pmatrix}$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 3 \\ 3.5 \end{pmatrix}$$

Extrapolation:

$$X(1/2) = \frac{4x(h = 1/2) - x(h = 1))}{3} = \frac{4 \cdot 3 - 3}{3} = 3$$

All solutions here are exact.

3. (7.1-2:a3) Solve the boundary value problem by shooting method:
72.2

$$x'' - 3x' + 2x = t + 1,$$
$$x(0) = 1, \quad x(1) = 3$$

(a) Find the exact solution. $(x_H + x_P)$

(b) Find the exact solution $u = x(t)$ for (shooting 1)

$$x'' - 3x' + 2x = t + 1, \qquad x(0) = 1, \qquad x'(0) = 0$$

(c) Find the exact solution $v = x(t)$ for (shooting 2)

$$x'' - 3x' + 2x = 0 \qquad x(0) = 0, \qquad x'(0) = 1$$

(d) Then combine u and v by the shooting method to get the exact solution.

(e) Convert the above two shooting problems to systems of two equations, and apply the Euler method with $h = 1/2$ to them. Combine the two discrete solutions and find the approximate value of $x(1/2)$. Find the error.

(f) Solve the BVP by the Euler shooting method with $h = 1/4$. Find the error at $t = 1/2$.

(g) Solve the BVP by the Runge-Kutta shooting method with $h = 1/2$. Find the error at $t = 1/2$.

(h) Solve the finite difference equations for the boundary value problem with grid size $h = 1/2$ to get an approximation of $x(1/2)$. Find the error.

(i) Solve the BVP by the finite difference method with $h = 1/4$. Find the error at $t = 1/2$. Use extrapolation on the two $x(1/2)$ values to find a new $x(1/2)$ and the error.

- **ans:**

(a) $r^2 - 3r + 2 = 0$, $r = 1, 2$, $x_H = C_1 e^t + C_2 e^{2t}$,

$$x_P = Bt + C \Rightarrow 0 - 3(B) + 2(Bt + C) = t + 1$$
$$\Rightarrow B = \frac{1}{2},\ C = \frac{5}{4}$$

$$x = x_H + x_P = C_1 e^t + C_2 e^{2t} + \frac{t}{2} + \frac{5}{4}.$$

By boundary conditions,

$$1 = C_1 + C_2 + \frac{5}{4}$$
$$3 = C_1 e + C_2 e^2 + \frac{9}{4}.$$
$$x = -\frac{5 + e}{4(e^2 - e)} e^t + \frac{5 + e^2}{4(e^2 - e)} e^{2t} + \frac{t}{2} + \frac{5}{4}.$$

$$x(\frac{1}{2}) = 1.5296.$$

(b) As above,

$$u = x_H + x_P = C_1 e^t + C_2 e^{2t} + \frac{t}{2} + \frac{5}{4}.$$

By initial conditions,

$$1 = x(0) = C_1 + C_2 + \frac{5}{4}$$
$$0 = x'(0) = C_1 + 2C_2 + \frac{1}{2}$$

$$u = -\frac{1}{4} e^{2t} + \frac{t}{2} + \frac{5}{4}.$$

(c) As before

$$v = x = x_H = C_1 e^t + C_2 e^{2t}.$$

By initial conditions,

$$0 = C_1 + C_2$$
$$1 = C_1 + 2C_2$$

$$v = -e^t + e^{2t}.$$

(d) Combining u and v, we need to find

$$c = \frac{x(1) - u(1)}{v(1)} = \frac{3 + \frac{1}{4}e^2 - \frac{1}{2} - \frac{5}{4}}{-e + e^2} = \frac{\frac{1}{4}e^2 + \frac{5}{4}}{-e + e^2}$$

$$x = u + cv = -\frac{5 + e}{4(e^2 - e)} e^t + \frac{5 + e^2}{4(e^2 - e)} e^{2t} + \frac{t}{2} + \frac{5}{4}.$$

(e) <u>For u:</u>

$$x' = y$$
$$y' = 3y - 2x + t + 1$$

$h = 1/2$, $U = Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ $U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$U' = F(t, U) = \begin{pmatrix} y \\ 3y - 2x + t + 1 \end{pmatrix}$$

$$\begin{aligned} U_1 &= U_0 + hF(t_0, U_0) \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 3(0) - 2(1) + 0 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} U_2 &= U_1 + hF(t_1, U_1) \\ &= \begin{pmatrix} .75 \\ -1.5 \end{pmatrix} \end{aligned}$$

$$u(1) \sim (U_2)_1 = .75$$

<u>For v:</u>

$$\begin{aligned} x' &= y \\ y' &= 3y - 2x \end{aligned}$$

$h = 1/2$,

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V' = F(t, V) = \begin{pmatrix} y \\ 3y - 2x \end{pmatrix}$$

$$\begin{aligned} V_1 &= V_0 + hF(t_0, V_0) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 3(1) - 2(0) \end{pmatrix} = \begin{pmatrix} .5 \\ 2.5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} V_2 &= V_1 + hF(t_1, V_1) \\ &= \begin{pmatrix} 1.75 \\ 5.75 \end{pmatrix} \end{aligned}$$

$$v(1) \sim (V_2)_1 = 1.75$$

Combine them:

$$\begin{aligned} x(t) &= u + \frac{\beta - u(b)}{v(b)} v \\ &= u + \frac{3 - .75}{1.75} v \\ &= u + 1.2857v \end{aligned}$$

In particular,

$$\begin{aligned} x(1/2) &\sim u(1/2) + 1.2857v(1/2) = (U_1)_1 + 1.2857(V_1)_1 \\ &= 1.64285 \end{aligned}$$

Error is -0.1131.

(f) <u>For u:</u> $h = 1/4$, $U = Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$ $U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$U' = F(t, U) = \begin{pmatrix} y \\ 3y - 2x + t + 1 \end{pmatrix}$$

$$\begin{aligned} U_1 &= U_0 + hF(t_0, U_0) \\ &= \begin{pmatrix} 1 \\ -.25 \end{pmatrix} \end{aligned}$$

$$U_2 = U_1 + hF(t_1, U_1) = \begin{pmatrix} 0.9375 \\ -0.625 \end{pmatrix}$$

$$U_3 = U_2 + hF(t_2, U_2) = \begin{pmatrix} 0.78125 \\ -1.1875 \end{pmatrix}$$

$$U_4 = U_4 + hF(t_3, U_3) = \begin{pmatrix} 0.484375 \\ -2.03125 \end{pmatrix}$$

$$u(1) \sim (U_4)_1 = .484375$$

<u>For v:</u>

$$\begin{aligned} x' &= y \\ y' &= 3y - 2x \end{aligned}$$

$h = 1/4$,

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V' = F(t, V) = \begin{pmatrix} y \\ 3y - 2x \end{pmatrix}$$

$$\begin{aligned} V_1 &= V_0 + hF(t_0, V_0) \\ &= \begin{pmatrix} .25 \\ 1.75 \end{pmatrix} \end{aligned}$$

$$V_2 = V_1 + hF(t_1, V_1) = \begin{pmatrix} 0.6875 \\ 2.9375 \end{pmatrix}$$

$$V_3 = V_2 + hF(t_2, V_2) = \begin{pmatrix} 1.421875 \\ 4.796875 \end{pmatrix}$$

$$V_4 = V_3 + hF(t_3, V_3) = \begin{pmatrix} 2.62109375 \\ 7.68359375 \end{pmatrix}$$

$$v(1) \sim (V_4)_1 = 2.62109375$$

Combine them:

$$\begin{aligned} x(t) &= u + \frac{\beta - u(b)}{v(b)} v \\ &= u + \frac{3 - .484375}{2.62109375} v \\ &= u + .95976v \end{aligned}$$

In particular,

$$\begin{aligned} x(1/2) &\sim u(1/2) + .95976v(1/2) \\ &= (U_2)_1 + .95976(V_2)_1 \\ &= .9375 + .95976 * (0.6875) = 1.597355 \end{aligned}$$

Error is -0.0676. (smaller than that of grid size 1/2 case)

(g) For u: Same system as above.

$$\begin{aligned} k_1 &= F(t_0, U_0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ k_2 &= F(t_0 + \frac{h}{2}, U_0 + \frac{h}{2}k_1) = \begin{pmatrix} -.25 \\ -1.5 \end{pmatrix} \\ k_3 &= F(t_0 + \frac{h}{2}, U_0 + \frac{h}{2}k_2) = \begin{pmatrix} -0.375 \\ -1.75 \end{pmatrix} \\ k_4 &= F(t_0 + h, U_0 + hk_3) = \begin{pmatrix} -0.875 \\ -2.75 \end{pmatrix}; \\ U_1 &= U_0 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\ &= \begin{pmatrix} 0.8229 \\ -0.85416 \end{pmatrix}; \\ t_1 &= 0.5 \\ k_1 &= F(t_1, U_1) = \begin{pmatrix} -0.8541666 \\ -2.70833 \end{pmatrix} \\ k_2 &= F(t_1 + \frac{h}{2}, U_1 + \frac{h}{2}k_1) = \begin{pmatrix} -1.53125 \\ -4.0625 \end{pmatrix} \\ k_3 &= F(t_1 + \frac{h}{2}, U_1 + \frac{h}{2}k_2) = \begin{pmatrix} -1.869791 \\ -4.739583 \end{pmatrix} \\ k_4 &= F(t_1 + h, U_1 + hk_3) = \begin{pmatrix} -3.22395 \\ -7.4479 \end{pmatrix}; \\ U_2 &= U_1 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\ &= \begin{pmatrix} -0.08376736 \\ -3.16753472 \end{pmatrix}; \end{aligned}$$

$$u(1) \sim (U_2)_1 = -0.08376736$$

For v:

$$\begin{aligned} x' &= y \\ y' &= 3y - 2x \end{aligned}$$

$h = 1/2$,

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V' = F(t, V) = \begin{pmatrix} y \\ 3y - 2x \end{pmatrix}$$

$$V' = F(t, V) = \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}$$

$$\begin{aligned} k_1 &= F(t_0, V_0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ k_2 &= F(t_0 + \frac{h}{2}, V_0 + \frac{h}{2}k_1) = \begin{pmatrix} 1.75 \\ 4.75 \end{pmatrix} \\ k_3 &= F(t_0 + \frac{h}{2}, V_0 + \frac{h}{2}k_2) = \begin{pmatrix} 2.1875 \\ 5.6875 \end{pmatrix} \\ k_4 &= F(t_0 + h, V_0 + hk_3) = \begin{pmatrix} 3.84375 \\ 9.34375 \end{pmatrix}; \\ V_1 &= V_0 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\ &= \begin{pmatrix} 1.05989583 \\ 3.7682291 \end{pmatrix}; \\ t_1 &= 0.5 \\ k_1 &= F(t_1, V_1) = \begin{pmatrix} 3.7682291 \\ 9.18489583 \end{pmatrix} \\ k_2 &= F(t_1 + \frac{h}{2}, V_1 + \frac{h}{2}k_1) = \begin{pmatrix} 6.064453125 \\ 14.189453125 \end{pmatrix} \\ k_3 &= F(t_1 + \frac{h}{2}, V_1 + \frac{h}{2}k_2) = \begin{pmatrix} 7.315592 \\ 16.794759 \end{pmatrix} \\ k_4 &= F(t_1 + h, V_1 + hk_3) = \begin{pmatrix} 12.16560872 \\ 27.06144205 \end{pmatrix}; \\ V_2 &= V_1 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\ &= \begin{pmatrix} 4.6177232 \\ 11.9527926 \end{pmatrix}; \end{aligned}$$

$$v(1) \sim (V_2)_1 = 4.6177232$$

Combine them:

$$\begin{aligned} x(t) &= u + \frac{\beta - u(b)}{v(b)}v \\ &= u + \frac{3 - (-0.08376736)}{4.6177232}v \\ &= u + 0.66781121174427v \end{aligned}$$

In particular,

$$\begin{aligned} x(1/2) &\sim u(1/2) + 0.66781121174427v(1/2) \\ &= (U_1)_1 + 0.66781121174427(V_1)_1 \\ &= 0.8229 + 0.66781121174427 * 1.05989583333333 \\ &= 1.530710 \end{aligned}$$

Error is -0.001. (much smaller)

(h) Central difference:

$$x_1'' = \frac{x_0 - 2x_1 + x_2}{h^2}$$

$$x_1' = \frac{x_2 - x_0}{2h}$$

Only 1 unknown x_1. $x_0 = 1$, and $x_2 = 3$. At $t = t_1 = 1/2$,

$$\frac{x_0 - 2x_1 + x_2}{h^2} - 3\frac{x_2 - x_0}{2h} + 2x_1 = \frac{1}{2} + 1$$

$$
\begin{aligned}
& -2x_1 + 2h^2 x_1 \\
=& h^2(\frac{3}{2}) - x_0 - x_2 - 3\frac{x_0 h}{2} + 3\frac{x_2 h}{2} \\
-1.5x_1 =& -2.125 \\
\Rightarrow & x(1/2) \sim x_1 = 1.416666
\end{aligned}
$$

The error is 0.113.

(i) 3 unknowns x_1, x_2, x_3. Given $x_0 = 1$, and $x_4 = 3$. At $t = t_1 = 1/4$,

$$\frac{x_0 - 2x_1 + x_2}{h^2} - 3\frac{x_2 - x_0}{2h} + 2x_1 = \frac{1}{4} + 1$$

$$
\begin{aligned}
& -2x_1 + x_2 - 3\frac{x_2 h}{2} + 2h^2 x_1 \\
=& h^2(\frac{1}{4} + 1) - x_0 - 3\frac{x_0 h}{2}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& x_1 - 2x_2 + x_3 - 3\frac{x_3 h}{2} + 3\frac{x_1 h}{2} + 2h^2 x_2 \\
=& h^2(\frac{1}{2} + 1) \\
& x_2 - 2x_3 + x_3 + 3\frac{x_2 h}{2} + 2h^2 x_2 \\
=& h^2(\frac{3}{4} + 1) - x_4 + 3\frac{x_4 h}{2}
\end{aligned}
$$

Write three equations in a system $Ax = b$:

$$\begin{pmatrix} -1.875 & 0.625 & \\ 1.375 & -1.875 & 0.625 \\ & 1.375 & -1.875 \end{pmatrix} x = \begin{pmatrix} -1.296875 \\ 0.09375 \\ -1.765625 \end{pmatrix}$$

$$x = A^{-1}b = \begin{pmatrix} 1.1945 \\ 1.508 \\ 2.0480 \end{pmatrix}$$

$$x(1/2) \sim x_2 = 1.508$$

The error is 0.02097 (about 1/4 of the last error because the method is of second order and the grid size is halved).

4. (7.2:a5) Solve the boundary value problem (finding
72.41 $x(1)$):

$$x'' + x = e^x, \qquad x(0) = 1, \quad x(2) = 4$$

(a) by the Euler **nonlinear** shooting method with $h = 1$ and 2 steps (5 shootings) of the bisection method on $x'(0) \in [0, 2]$.

(b) by the finite difference method with $h = 1$; with 4 steps of the Newton's method starting with $x_1 = 2$.

• **ans:** Convert the second order equation to a first order system:

$$
\begin{aligned}
x' &= y \\
y' &= x'' = -x + e^x
\end{aligned}
$$

$$Z = \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$Z' = F(t, Z) = \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} y \\ -x + e^x \end{pmatrix}$$

Discretization:

$$h = 1, \; t_0 = 0, \; t_1 = 1, \; t_2 = 2$$

Shooting 1: $x' = 0$ (the lower end point)

$$
\begin{aligned}
Z_0 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
Z_1 &= Z_0 + hF(t_0, Z_0) \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1.7183 \end{pmatrix} \\
&= \begin{pmatrix} 1.0000 \\ 1.7183 \end{pmatrix} \\
Z_2 &= Z_1 + hF(t_1, Z_1) \\
&= \begin{pmatrix} 2.7183 \\ 3.4366 \end{pmatrix}
\end{aligned}
$$

$x(2) \sim x_2 = Z_{2,1} = 2.7183$. We shoot too low (target $x(2) = 4$).

Shooting 2: $x' = 2$ (the high end point)

$$
\begin{aligned}
Z_0 &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
Z_1 &= Z_0 + hF(t_0, Z_0) \\
&= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1.7183 \end{pmatrix} \\
&= \begin{pmatrix} 3.0000 \\ 3.7183 \end{pmatrix} \\
Z_2 &= Z_1 + hF(t_1, Z_1) \\
&= \begin{pmatrix} 6.7183 \\ 20.8038 \end{pmatrix}
\end{aligned}
$$

$x(2) \sim x_2 = Z_{2,1} = 6.7183$. We shoot too high (target $x(2) = 4$).

Shooting 3: $x' = 1$ (the middle point)

$$
\begin{aligned}
Z_0 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
Z_1 &= Z_0 + hF(t_0, Z_0) \\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1.0000 \\ 1.7183 \end{pmatrix} \\
&= \begin{pmatrix} 2.0000 \\ 2.7183 \end{pmatrix} \\
Z_2 &= Z_1 + hF(t_1, Z_1) \\
&= \begin{pmatrix} 4.7183 \\ 8.1073 \end{pmatrix}
\end{aligned}
$$

$x(2) \sim x_2 = Z_{2,1} = 4.7183$. We shoot too high. What do we try next? Between $(0, 1)$ or between $(1, 2)$?

Shooting 4: $x' = \frac{0+1}{2} = 0.5$ (the middle point of $(1, 2)$)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 0.5000 \\ 1.7183 \end{pmatrix} \\ &= \begin{pmatrix} 1.5000 \\ 2.2183 \end{pmatrix} \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= \begin{pmatrix} 3.7183 \\ 5.2000 \end{pmatrix} \end{aligned}$$

$x(2) \sim x_2 = Z_{2,1} = 3.7183$. We shoot too low. What do we try next? Between $(0, 0.5)$ or between $(0.5, 1)$?

Shooting 5: $x' = \frac{0.5+1}{2} = 0.75$ (the middle point of $(1, 1.5)$)

$$\begin{aligned} Z_0 &= \begin{pmatrix} 1 \\ 0.75 \end{pmatrix} \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 1 \\ 0.75 \end{pmatrix} + \begin{pmatrix} 0.7500 \\ 1.7183 \end{pmatrix} \\ &= \begin{pmatrix} 1.7500 \\ 2.4683 \end{pmatrix} \\ Z_2 &= Z_1 + hF(t_1, Z_1) \\ &= \begin{pmatrix} 4.2183 \\ 6.4729 \end{pmatrix} \end{aligned}$$

$x(2) \sim x_2 = Z_{2,1} = 4.2183$. We shoot too high (target $x(2) = 20$). The next try for x' is between $(0.5, 0.75)$.

Final answer: Read from last computation:

$$x(1) \simeq Z_{1,1} = 1.75$$

Finite element method.

$$x_i'' \simeq \frac{x_{i-1} - 2x_i + x_{i+1}}{h^2}$$

$$x_i' \simeq \frac{x_{i+1} - x_{i-1}}{2h}$$

$$h = 1,\ t_0 = 0,\ t_1 = 1,\ t_2 = 2$$

We have only 1 unknown x_1.

$$\frac{x(0) - 2x_1 + x(2)}{h^2} + x_1 = e^{x_1}$$

$$\frac{1 - 2x_1 + 4}{1^2} + x_1 = e^{x_1}$$

$$f(x_1) = 5 - x_1 - e^{x_1} = 0$$

$$f'(x_1) = -1 - e^{x_1}$$

$$\begin{aligned} p_0 &= 2; \\ p_{i+1} &= p_i - f(p_i)/f'(p_i) \\ p_1 &= 2 - (-4.3891)/(-8.3891) \\ &= 1.4768 \\ p_2 &= 1.3177 \\ p_3 &= 1.3066 \\ p_4 &= 1.3066 \end{aligned}$$

Final answer (for finite difference method):

$$x(1) \simeq x_1 = 1.3066$$