M353 Hw 9 (S. Zhang) 6.1 6.3, 6.4.

1. 6.1:4, 6

66.99 1. (6.1:4c,6.1:6c) Solve the IVP. Find $y(1)$, Apply Euler
66.21 method with $h = 1/4$. Find errors.

$$y' = 4t - 2y; \quad y(0) = 0.$$

- **ans:**

First order linear equation

$$y' + 2y = 4t$$

Formula:

$$\boxed{y' + py = q \Rightarrow \mu = e^{\int p};\ y = \mu^{-1} \int \mu q}$$

$$\mu = e^{\int p} = e^{2t}$$

$$\begin{aligned} y =& \mu^{-1} \int \mu q = e^{-2t} \int e^{2t} 4t \\ =& e^{-2t}(e^{2t} 2t - e^{2t} + C) \\ =& Ce^{-2t} + 2t - 1 \end{aligned}$$

By initial condition,

$$y = e^{-2t} + 2t - 1$$

Exact

$$y(1) = 1.1353$$

$$y_{i+1} = y_i + hf_i = y_i + hf(t_i, y_i)$$

$$\begin{aligned} h &= \frac{1}{4}, t_0 = 0,\ y_0 = 0 \\ t_1 &= \frac{1}{4},\ y_1 = 0,\ err = 0.1065 \\ t_2 &= \frac{1}{2},\ y_2 = 0.2500,\ err = 0.1179 \\ t_3 &= \frac{3}{4},\ y_3 = 0.6250,\ err = 0.0981 \\ t_4 &= 1,\ y_4 = 1.0625,\ err = 0.0728 \end{aligned}$$

2. 6.3: 1cd, 3, 5, a1

67.99 1. (6.3:1c) Given an initial value problem
67.12

$$\begin{cases} x' = -y \\ y' = x \end{cases} \quad 0 < t < 1$$
$$\begin{cases} x(0) = 1 \\ y(0) = 0. \end{cases}$$

Use Euler's method with step size $h = 1/4$.

- **ans:** Let

$$Z = \begin{pmatrix} x \\ y \end{pmatrix}$$

The exact solution is

$$Z(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

(We do not need to find it. But if we have it, we can use it to do a rough checking of our numerical results.)
Apply the Euler method,

$$t_0 = 0; Z_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; h = 0.25;$$

$$Z' = F(t, Z) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$\begin{aligned} Z_i &= Z_{i-1} + hF(t_{i-1}, Z_{i-1}) \\ Z_1 &= Z_0 + hF(t_0, Z_0) \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + h \begin{pmatrix} 0 \\ 1.0000 \end{pmatrix} \\ &= \begin{pmatrix} 1.0000 \\ 0.2500 \end{pmatrix} \\ err = Z(t) - Z_1 &= \begin{pmatrix} -0.0311 \\ -0.0026 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Z_2 &= \begin{pmatrix} 1.0000 \\ 0.2500 \end{pmatrix} + h \begin{pmatrix} -0.2500 \\ 1.0000 \end{pmatrix} \\ &= \begin{pmatrix} 0.9375 \\ 0.5000 \end{pmatrix} \\ err &= \begin{pmatrix} -0.0599 \\ -0.0206 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Z_3 &= \begin{pmatrix} 0.9375 \\ 0.5000 \end{pmatrix} + h \begin{pmatrix} -0.5000 \\ 0.9375 \end{pmatrix} \\ &= \begin{pmatrix} 0.8125 \\ 0.7344 \end{pmatrix} \\ err &= \begin{pmatrix} -0.0808 \\ -0.0527 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Z_4 &= \begin{pmatrix} 0.8125 \\ 0.7344 \end{pmatrix} + h \begin{pmatrix} -0.7344 \\ 0.8125 \end{pmatrix} \\ &= \begin{pmatrix} 0.6289 \\ 0.9375 \end{pmatrix} \\ err &= \begin{pmatrix} -0.0886 \\ -0.0960 \end{pmatrix} \end{aligned}$$

(It is not required to find the errors here.)

2. (6.3:3) Convert a high order equation into a first order
67.14 system

$$a) \quad y'' - ty = 0$$
$$b) \quad y'' - 2ty' + 2y = 0$$
$$c) \quad y'' - ty' - y = 0$$

- **ans:** Let

$$Z = \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix}$$

$$a) \quad Z' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} z \\ ty \end{pmatrix}$$
$$b) \quad Z' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} z \\ 2tz - 2y \end{pmatrix}$$
$$c) \quad Z' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} z \\ tz + y \end{pmatrix}$$

3. (6.3:a1) Given an initial value problem
67.994

$$x'' - 3x' + 2x = 2e^{3t}, \quad 0 < t < 0.2$$
$$x(0) = 3, \quad x'(0) = 6.$$

Find the exact solution. Convert the equation to a system. Use Euler's method with
(1) $h = 0.1$ and find the error.
(2) $h = 0.05$ and find the error.
(3) Extrapolation and find the error.

- **ans:** We find solution in two steps. First, find x_H (for the homogeneous equation)

$$x'' - 3x' + 2x = 0$$
$$r^2 - 3r + 2 = 0, \; r = 1, 2$$
$$x_H = c_1 e^t + c_2 e^{2t}$$

Step 2, find a particular solution for the nonhomogeneous equation: Plug the unknown coefficient solution into the equation and get

$$x_p = Ae^{3t}, \; A = 1$$

$$x = x_H + x_p = c_1 e^t + c_2 e^{2t} + e^{3t}$$

By the initial condition, we get

$$x = e^t + e^{2t} + e^{3t}.$$

Next, we convert one equation to a system of first order equations. For a second order equation, we always let

$$y = x'$$

Then we put them together as a vector,

$$Z = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$Z' = \begin{pmatrix} y \\ 3y - 2x + 2e^{3t} \end{pmatrix}$$

Use Euler's method. (1)

$$t_0 = 0; Z_0 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}; h = 0.1;$$
$$Z_1 = Z_0 + hF(t_0, Z_0)$$
$$= \begin{pmatrix} 3.6 \\ 7.4 \end{pmatrix}$$

$$t_1 = t_0 + h = 0.1$$

$$Z_2 = Z_1 + hF(t_1, Z_1)$$
$$= \begin{pmatrix} 4.34 \\ 9.17 \end{pmatrix}$$
$$err = x(0.2) - Z_{2,1} = 0.1953$$

(2)

$$t_0 = 0; Z_0 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}; h = 0.05;$$
$$Z_1 = Z_0 + hF(t_0, Z_0)$$
$$= \begin{pmatrix} 3.3 \\ 6.7 \end{pmatrix}$$
$$t_1 = t_0 + h = 0.05$$
$$Z_2 = Z_1 + hF(t_1, Z_1)$$
$$= \begin{pmatrix} 3.6350 \\ 7.4912 \end{pmatrix}$$
$$t_2 = t_1 + h = 0.1$$
$$Z_3 = \begin{pmatrix} 4.0096 \\ 8.3863 \end{pmatrix}$$
$$t_3 = 0.15$$
$$Z_3 = \begin{pmatrix} 4.4289 \\ 9.4002 \end{pmatrix}$$
$$err = x(0.2) - Z_{2,1} = 0.1065$$

About 1/2 of last error.

(3)

$$Z_{extra} = \frac{2Z_{h=0.05} - Z_{h=0.1}}{2 - 1} = \begin{pmatrix} 4.5178 \\ 9.6304 \end{pmatrix}$$
$$err = x(0.2) - Z_{extra,1} = 0.0176$$

The error is much smaller (about 1/10).

If we repeat this extrapolation with the results of $h = 0.05$ and $h = 0.025$, the new error would be about 1/4, as we get a second order method after extraplation.

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1. 6.4: 3bcd, a1
69.99
1. (6.4:3b) Solve initial value problem
69.11

$$y' = t^2 y; \quad y(0) = 1.$$

Use the Runge-Kutta method with $h = 1/4$ to approximate $y(1)$. Find the error.

- **ans:** This is a separable equation.

$$\int \frac{dy}{y} = \int t^2 dt$$

$$\ln y = \frac{t^3}{3} + C, \ y = C_1 e^{t^3/3}$$

by $y(0) = 1$,

$$y = e^{t^3/3}$$

$$\begin{aligned} t_0 &= 0; y_0 = 1; h = 0.25 \\ k_1 &= f(t_0, y_0) = 0 \\ k_2 &= f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) \\ &= f(t_0 + \frac{h}{2}, 1) \\ &= 0.015625 \\ k_3 &= f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2) \\ &= f(t_0 + \frac{h}{2}, 1.001953125) \\ &= 0.015655517578125 \\ k_4 &= f(t_0 + h, y_0 + hk_3) \\ &= f(t_0 + h, 1.00391387939453) \\ &= 0.0627446174621 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\ &= y_0 + h(0.02088427543640) \\ &= 1.00522106885 \\ Err &= y(t_1) - y_1 = 0.00000085142 \end{aligned}$$

$$\begin{aligned} t_1 &= 0.25, \ i = 1 \\ k_1 &= f(t_i, y_i) = 0.062826316 \\ k_2 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1) \\ &= f(t_0 + \frac{h}{2}, 1.01307435) \\ &= 0.142463581 \\ k_3 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2) \\ &= f(t_i + \frac{h}{2}, 1.0230290165) \\ &= 0.1438634554 \\ k_4 &= f(t_i + h, y_i + hk_3) \\ &= f(t_i + h, 1.0411869327) \\ &= 0.260296733 \end{aligned}$$

$$\begin{aligned} y_{i+1} &= y_i + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\ &= y_i + h(0.1492961873684) \\ &= 1.04254511570 \\ Err &= y(t_{i+1}) - y_{i+1} \\ &= 0.00000178 \end{aligned}$$

$$\begin{aligned} t_2 &= 0.5, \ i = 2 \\ k_1 &= f(t_i, y_i) = 0.2606362 \\ k_2 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1) \\ &= f(t_0 + \frac{h}{2}, 1.07512465) \\ &= 0.41997056 \\ k_3 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2) \\ &= f(t_i + \frac{h}{2}, 1.0950414365) \\ &= 0.4277505611 \\ k_4 &= f(t_i + h, y_i + hk_3) \\ &= f(t_i + h, 1.1494827559) \\ &= 0.64658405024 \end{aligned}$$

$$\begin{aligned} y_{i+1} &= y_i + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\ &= y_i + h(0.4337770974) \\ &= 1.1509893900 \\ Err &= y(t_{i+1}) - y_{i+1} \\ &= 0.000003554 \end{aligned}$$

$$\begin{aligned} t_3 &= 0.75, \ i = 3 \\ k_1 &= f(t_i, y_i) = 0.64743153 \\ k_2 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1) \\ &= f(t_0 + \frac{h}{2}, 1.23191833) \\ &= 0.94318747259 \\ k_3 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2) \\ &= f(t_i + \frac{h}{2}, 1.2688878241) \\ &= 0.971492240 \\ k_4 &= f(t_i + h, y_i + hk_3) \\ &= f(t_i + h, 1.3938624501) \\ &= 1.39386245015 \end{aligned}$$

$$
\begin{aligned}
y_{i+1} &= y_i + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\
&= y_i + h(0.9784422346) \\
&= 1.3955999487 \\
Err &= y(t_{i+1}) - y_{i+1} \\
&= 0.00001247
\end{aligned}
$$

2. (6.4:a1) (1) Solve initial value problem (hint: separable,
69.21 $y = 3e^{t^2 - 4\cos(\pi t)}$)

$$y' = (2t + 4\pi \sin(\pi t))y; \quad y(2) = 3.$$

(2) Use the Euler method with $h = .1$ and $h = 0.05$ to approximate $y(2.1)$. Find the extrapolated solution and all errors.
(3) Use the Runge-Kutta method with $h = .1$ and $h = 0.05$ to approximate $y(2.1)$. Find the extrapolated solution and all errors.

- **ans:**

$$
\begin{aligned}
\int \frac{dy}{y} &= \int (2t - 4\pi \sin(\pi t))dt \\
\ln y &= t^2 - 4\cos \pi t + C \\
y &= C_1 e^{t^2 - 4\cos(\pi t)}
\end{aligned}
$$

By the initial condition,

$$y = 3e^{t^2 - 4\cos(\pi t)}$$

$$y(2.1) = 5.498010081936182$$

$$
\begin{aligned}
t_0 &= 2; y_0 = 3; h = 0.1 \\
k_1 &= f(t_0, y_0) = 12; \\
k_2 &= f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) = 21.8369284 \\
k_3 &= f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2) = 24.820377131 \\
k_4 &= f(t_0 + h, y_0 + hk_3) = 44.312528 \\
y_1 &= y_0 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\
&= 5.493785657992 \\
Err &= y(2.1) - y_1 = 0.0042244
\end{aligned}
$$

Again

$$
\begin{aligned}
t_0 &= 2; y_0 = 3; h = 0.05 \\
k_1 &= f(t_0, y_0) = 12; \\
k_2 &= f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) = 16.61862204 \\
k_3 &= f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2) = 17.2001003367 \\
k_4 &= f(t_0 + h, y_0 + hk_3) = 23.414070404 \\
y_1 &= y_0 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\
&= 3.858762626439531
\end{aligned}
$$

$$t_1 = t_0 + h = 2.05$$

$$
\begin{aligned}
k_1 &= f(t_1, y_1) = 23.4065342959; \\
k_2 &= f(t_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_1) = 31.47882061 \\
k_3 &= f(t_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_2) = 32.9083339150 \\
k_4 &= f(t_1 + h, y_1 + hk_3) = 44.49150381537 \\
y_1 &= y_0 + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \\
&= 5.4976988527859
\end{aligned}
$$

$$Err = y(2.1) - y_2 = 0.0003112$$

Extrapolation

$$
\begin{aligned}
y_{new} =& \frac{2^4 y_2 - y_1^{\text{old}}}{2^4 - 1} \\
=& 5.497959732438912 \\
Err = y(2.1) - y_{new} =& 0.000050
\end{aligned}
$$