M353 Hw 7 (S. Zhang) 5.2 5.3 5.4 .

1\.

60.99 5.2: a1, a2, a3, 1ab, 2ab, 3ab, 10

1\. (5.2:a1) Compute $\int_1^3 x^5\,dx$ by the Trapezoidal rule with
60.71 $m = 1$ and $m = 2$, and extrapolation. Find the error each case.

- **ans:**

$$\int_1^3 x^5\,dx = 121.3333$$

$m = 1$, $h = 2$:

$$\begin{aligned} T &= \frac{h}{2}(f_0 + f_1) \\ &= 1^5 + 3^5 = 244 \end{aligned}$$

error:

$$\int -T = -122.66667$$

$m = 2$, $h = 1$:

$$\begin{aligned} T &= \frac{h}{2}(f_0 + 2f_1 + f_2) \\ &= 154 \end{aligned}$$

error:

$$\int -T = -32.66667$$

Extrapolation

$$extr = \frac{4T_{h=1} - T_{h=2}}{4-1} = 124$$

error:

$$\int -extr = -2.66667$$

2\. (5.2:a2) Compute $\int_1^3 x^5\,dx$ by the mid-point rule with
60.72 $m = 1$ and $m = 2$, and extrapolation. Find the error each case.

- **ans:**

$$\int_1^3 x^5\,dx = 121.3333$$

$m = 1$, $h = 2$:

$$\begin{aligned} M &= h(f_{1/2}) \\ &= 2(2^5) = 64 \end{aligned}$$

error:

$$\int -M = 57.3333$$

$m = 2$, $h = 1$:

$$\begin{aligned} M &= h(f_{1/2} + f_{3/2}) \\ &= 105.25 \end{aligned}$$

error:

$$\int -M = 16.0833$$

Extrapolation

$$extr = \frac{4M_{h=1} - M_{h=2}}{4-1} = 119$$

error:

$$\int -extr = 2.3333$$

3\. (5.2:a3) Compute $\int_1^3 x^5\,dx$ by the Simpson's rule with
60.73 $m = 1$ ($h = 1!$) and $m = 2$, and extrapolation. Find the error each case.

- **ans:**

$$\int_1^3 x^5\,dx = 121.3333$$

$m = 1$, $h = 1$:

$$\begin{aligned} S &= \frac{h}{3}(f_0 + 4f_1 + f_2) \\ &= \frac{1}{3}(1^5 + 4 \cdot 2^5 + 3^5) = 124 \end{aligned}$$

error:

$$\int -S = -2.66667$$

$m = 2$, $h = 1/2$:

$$\begin{aligned} S &= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) \\ &= 121.5 \end{aligned}$$

error:

$$\int -S = -0.166667$$

Extrapolation

$$extr = \frac{2^4 S_{h=1} - S_{h=2}}{2^4 - 1} = 121.3333$$

error:

$$\int -extr = 0$$

4. (5.2:1a) Compute $\int_0^1 x^2\,dx$ by the Trapezoidal rule with
60.74 $m = 1$, $m = 2$ and $m = 4$. Find the error each case.

- **ans:**

$$\int_1^3 x^2\,dx = 1/3$$

$m = 1$, $h = 1$:

$$\begin{aligned} T &= \frac{h}{2}(f_0 + f_1) \\ &= \frac{1}{2}(0+1) = .5 \end{aligned}$$

error:

$$\int -T = -0.1667$$

$m = 2$, $h = 1/2$:

$$\begin{aligned} T &= \frac{h}{2}(f_0 + 2f_1 + f_2) \\ &= 0.375 \end{aligned}$$

error:

$$\int -T = -0.0417$$

$m = 4$, $h = 1/4$:

$$\begin{aligned} T &= \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + f_4) \\ &= 0.3438 \end{aligned}$$

error:

$$\int -T = -.0104$$

5. (5.2:2a) Compute $\int_0^1 x^2\,dx$ by the midpoint rule with
60.76 $m = 1$, $m = 2$ and $m = 4$. Find the error each case.

- **ans:**

$$\int_1^3 x^2\,dx = 1/3$$

$m = 1$, $h = 1$:

$$\begin{aligned} M &= h(f_{1/2}) \\ &= (1/2)^2 = .25 \end{aligned}$$

error:

$$\int -M = 0.0833$$

$m = 2$, $h = 1/2$:

$$\begin{aligned} M &= h(f_{1/2} + f_{3/2}) \\ &= h(f(1/4) + f(3/4)) = 0.3125 \end{aligned}$$

error:

$$\int -M = 0.0208$$

$m = 4$, $h = 1/4$:

$$\begin{aligned} M &= h(f_{1/2} + f_{3/2} + f_{5/2} + f_{7/2}) \\ &= 0.3281 \end{aligned}$$

error:

$$\int -M = 0.0052$$

6. (5.2:3a) Compute $\int_0^1 x^2\,dx$ by the Simpson's rule with
60.78 $m = 1$, $m = 2$ and $m = 4$. Find the error each case.

- **ans:**

$$\int_1^3 x^2\,dx = 1/3$$

$m = 1$, $h = 1/2$:

$$\begin{aligned} S &= \frac{h}{3}(f_0 + 4f_1 + f_2) \\ &= \frac{h}{3}(f(0) + 4f(\frac{1}{2}) + f(1)) = \frac{1}{3} \end{aligned}$$

error:

$$\int -S = 0$$

$m = 2$, $h = 1/4$:

$$\begin{aligned} S &= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4) \\ &= \frac{h}{3}(f(0) + 4f(\frac{1}{4}) + 2f(\frac{1}{2}) + 4f(\frac{3}{4}) + f(1)) \\ &= \frac{1}{3} \end{aligned}$$

error:

$$\int -S = 0$$

$m = 4$, $h = 1/8$:

$$\begin{aligned} S &= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + ... + f_8) \\ &= \frac{1}{3} \end{aligned}$$

error:

$$\int -S = 0$$

7. (5.2:10) Find the constants so that
60.81

$$\int_0^1 f(x)\,dx = c_1 f(0) + c_2 f(\frac{1}{2}) + c_3 f(1)$$

is exact for P_2 polynomials. What rule do you get?

- **ans:** Let $f(x) = 1,\ x, x^2$, we get 3 equations:

$$1 = \int_0^1 dx = c_1 + c_2 + c_3$$
$$\frac{1}{2} = \int_0^1 x\,dx = c_1(0) + c_2(\frac{1}{2}) + c_3(1)$$
$$\frac{1}{3} = \int_0^1 x^2\,dx = c_1(0)^2 + c_2(\frac{1}{2})^2 + c_3(1)^2$$

3 equations and 3 unknowns, we get

$$c_1 = c_3 = \frac{1}{6},\ c_2 = \frac{2}{3}$$

That is

$$\int_0^1 f(x)\,dx = \frac{f(0)}{6} + \frac{2}{3}f(\frac{1}{2}) + \frac{f(1)}{6}$$

This is the Simpson's rule.

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1. 5.3: 1ab, a1-a2
62.99

1. (5.3:1a) Find Romberg R_{33} for
62.26

$$\int_0^1 x^2\,dx.$$

- **ans:**

$$R_{11} = T_{m=1} = h[\frac{f_0}{2} + \frac{f_1}{2}] = 0.5000$$
$$R_{21} = T_{m=2} = \frac{R_{11}}{2} + hf_1 = 0.3750$$
$$R_{31} = T_{m=4} = \frac{R_{21}}{2} + h[f_1 + f_3] = 0.3438$$
$$R_{22} = \frac{4R_{21} - R_{11}}{3} = \frac{1}{3}$$
$$R_{32} = \frac{4R_{31} - R_{21}}{3} = \frac{1}{3}$$
$$R_{33} = \frac{16R_{32} - R_{22}}{16-1} = \frac{1}{3}$$

Exact

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

2. (5.3:a1) Find (1) $T_{h=1}$, (2) $T_{h=1/2}$ by the recursive formula, (3) $T_{h=1/4}$ by the recursive formula, (4) Romberg R_{33} and (5) Simpson $S_{h=1/2}$ for
62.29

$$\int_1^2 8x^3\,dx.$$

- **ans:**

$$R_{11} = T_{h=1} = T_{m=1} = h[\frac{f_0}{2} + \frac{f_1}{2}] = 36$$
$$R_{21} = T_{h=1/2} = T_{m=2} = \frac{R_{11}}{2} + hf_1 = 31.5$$
$$R_{31} = T_{h=1/4} = T_{m=4} = \frac{R_{21}}{2} + h[f_1 + f_3] = 30.375$$
$$R_{22} = \frac{4R_{21} - R_{11}}{3} = 30$$
$$R_{32} = \frac{4R_{31} - R_{21}}{3} = 30$$
$$R_{33} = \frac{16R_{32} - R_{22}}{16-1} = 30$$

$$S_{h=1/2} = \frac{4}{3}T_{h=1/2} + \frac{1}{3}T_{h=1}$$
$$= \frac{2}{3}(31.5) + \frac{1}{3}(36) = 30$$

Exact

$$\int_1^2 8x^3\,dx = 30$$

3. (5.3:a2) Suppose we are given
62.291

$$T_{h=8} = 20,\ T_{h=4} = 10,\ T_{h=2} = 6,\ T_{h=1} = 4$$

find the Romberg integral R_{44}.

- **ans:**

$$R_{11} = T_{h=8} = 20$$
$$R_{21} = T_{h=4} = 10$$
$$R_{31} = T_{h=2} = 6$$
$$R_{41} = T_{h=2} = 4$$
$$R_{22} = \frac{4R_{21} - R_{11}}{3} = \frac{20}{3} = 6.67$$
$$R_{32} = \frac{4R_{31} - R_{21}}{3} = \frac{14}{3} = 4.67$$
$$R_{42} = \frac{4R_{41} - R_{31}}{3} = \frac{10}{3} = 3.33$$
$$R_{33} = \frac{2^4 R_{32} - R_{22}}{2^4 - 1} = \frac{68}{15} = 4.5333$$
$$R_{43} = \frac{2^4 R_{42} - R_{32}}{2^4 - 1} = \frac{146}{15} = 3.2444$$
$$R_{44} = \frac{2^6 R_{43} - R_{33}}{2^6 - 1} = \frac{403}{125} = 3.224$$

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1. 5:4:1ab
63.99

1. (5.4:1b) Compute the integral by adaptive Trapezoid quadrature with tolerance 0.05
63.12

$$\int_0^{\pi/2} \cos x\,dx$$

- **ans:**

(a) On $(0, \pi/2)$,

$$T_{m=1} = 0.7854, \; T_{m=2} = 0.9481$$
$$tol = 3 \cdot TOL \cdot \frac{b_1 - a_1}{b - a} = 0.15$$
$$T_{m=1} - T_{m=2} = -0.1627$$

Not accepted. We cut the interval into 2.

(b) On $(0, \pi/4)$,

$$T_{m=1} = 0.6704,$$
$$T_{m=2} = 0.6980$$
$$tol = 3 \cdot TOL \cdot \frac{b_1 - a_1}{b - a} = 0.075$$
$$T_{m=1} - T_{m=2} = -0.0276$$

Accepted.

(c) On $(\pi/4, \pi/2)$,

$$T_{m=1} = 0.2777,$$
$$T_{m=2} = 0.2891$$
$$tol = 3 \cdot TOL \cdot \frac{b_1 - a_1}{b - a} = 0.075$$
$$T_{m=1} - T_{m=2} = -0.0114$$

Accepted.

(d) Together

$$T = 0.6980 + 0.2891 = 0.9871$$

$$\int_0^{\pi/2} \cos x \, dx = 1$$

Error

$$\int -T = 2 - 2.0547 = 0.0129$$

It is less than 0.05
