M353 Hw 6 (S. Zhang) 3.5,4.1,5.1.

1. 3.5:1, 2

26.99

1. (3.5:1c) Find the Bezier curve given 4 points:

26.23

$$\mathbf{x}_i = \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}$$

- **ans:**

$$\mathbf{a} = \mathbf{x}_1 = \begin{pmatrix}1\\2\end{pmatrix}$$
$$\mathbf{b} = 3(\mathbf{x}_2 - \mathbf{x}_1) = \begin{pmatrix}0\\3\end{pmatrix}$$
$$\mathbf{c} = 3(\mathbf{x}_3 - \mathbf{x}_2) - \mathbf{b} = \begin{pmatrix}3\\-3\end{pmatrix}$$
$$\mathbf{d} = \mathbf{x}_4 - \mathbf{x}_1 - \mathbf{b} - \mathbf{c} = \begin{pmatrix}-2\\0\end{pmatrix}.$$

$$\mathbf{P}_3(t) = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}0\\3\end{pmatrix}t + \begin{pmatrix}3\\-3\end{pmatrix}t^2 + \begin{pmatrix}-2\\0\end{pmatrix}t^3$$

or

$$x(t) = 1 + 3t^2 - 2t^3$$
$$y(t) = 2 + 3t - 3t^2$$

check:

$$\mathbf{P}(0) = \mathbf{x}_1,\ \mathbf{P}'(0) = 3(\mathbf{x}_2 - \mathbf{x}_1),$$
$$\mathbf{P}(1) = \mathbf{x}_4,\ \mathbf{P}'(1) = 3(\mathbf{x}_4 - \mathbf{x}_3)$$

2. (3.5:2c) Find the $\mathbf{x}_i$ for the Bezier Curve

26.26

$$x(t) = 2 + t^2 - t^3$$
$$y(t) = 1 - t + 2t^3$$

- **ans:** Method 1:

$$\mathbf{x}_1\mathbf{P}(0),\ \mathbf{x}_2 = \mathbf{P}'(0)/3 + \mathbf{x}_1,$$
$$\mathbf{x}_4 = \mathbf{P}(1),\ \mathbf{x}_3 = \mathbf{x}_4/\mathbf{P}'(1)$$

Method 2:

$$\mathbf{x}_1 = \mathbf{a} = \begin{pmatrix}2\\1\end{pmatrix}$$
$$\mathbf{x}_2 = \mathbf{b}/3 + \mathbf{x}_1 = \begin{pmatrix}2 & 2/3\end{pmatrix}$$
$$\mathbf{x}_3 = \mathbf{b}/3 + \mathbf{c}/3 + \mathbf{x}_2 = \begin{pmatrix}7/3\\1/3\end{pmatrix}$$
$$\mathbf{x}_4 = \mathbf{d} + \mathbf{x}_1 + \mathbf{b} + \mathbf{c} = \begin{pmatrix}2\\2\end{pmatrix}$$

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1. 4.1: a1, a2, a3, 2, 8a, 12

27.99

1. (4.1:a1) Find the least-squares solution for the over-

27.41 determined system. Compare the residual for the least-squares solution with "solutions" (1) $(0,0)$, (2) $(1,1)$.

$$\begin{pmatrix}1 & 1\\1 & 0\\2 & -3\end{pmatrix} x = \begin{pmatrix}-1\\2\\1\end{pmatrix}.$$

- **ans:**

$$A^T A x = A^T b$$

$$A^T A = \begin{pmatrix}6 & -5\\-5 & 10\end{pmatrix}$$

$$(A^T A)^{-1} = \begin{pmatrix}0.28571 & 0.14286\\0.14286 & 0.17143\end{pmatrix}$$

$$A^T b = \begin{pmatrix}3\\-4\end{pmatrix}$$

$$x = (A^T A)^{-1} A^T b = \begin{pmatrix}0.2857\\-0.2571\end{pmatrix}$$

$$r = b - Ax = \begin{pmatrix}-1.0286\\1.7143\\-0.3429\end{pmatrix},$$
$$\|b - Ax\|_2 = \sqrt{1.0286^2 + ...} = 2.0284$$
$$r1 = b - A\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}-1\\2\\1\end{pmatrix}$$
$$\|r1\|_2 = 2.4495 \text{ bigger}$$
$$r2 = b - A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix}$$
$$\|r2\|_2 = 3.7417 \text{ bigger}$$

2. (4.1:a2) Find the least-squares solution for the under-

27.42 determined system. Compare the length for the least-squares solution with solutions (1) $(14,0,28)$, (2) $(0,70/3,70/3)$, (3) a solution found by Gauss elimination.

$$\begin{pmatrix}1 & 1 & 2\\1 & 0 & -3\end{pmatrix} x = \begin{pmatrix}70\\-70\end{pmatrix}.$$

- **ans:**

$$yA^T x$$

$$AA^T y = b$$

$$AA^T = \begin{pmatrix}6 & -5\\-5 & 10\end{pmatrix}$$

$$(AA^T)^{-1} = \begin{pmatrix} 0.28571 & 0.14286 \\ 0.14286 & 0.17143 \end{pmatrix}$$

$$y = (AA^T))^{-1}b = \begin{pmatrix} 10 \\ -2 \end{pmatrix}$$

$$x = A^T y = \begin{pmatrix} 8 \\ 10 \\ 26 \end{pmatrix}$$

$$r = b - Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\|x\|_2 = \sqrt{8^2 + 10^2 + 26^2} = 28.98$$

$$r_1 = b - A \begin{pmatrix} 14 \\ 0 \\ 28 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\|x_1\|_2 = \sqrt{14^2 + 0^2 + 28^2} = 31.3 \text{ bigger}$$

$$r_2 = b - A \begin{pmatrix} 0 \\ 70/3 \\ 70/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\|x_2\|_2 = 32.99 \text{ bigger}$$

3. (4.1:a3) Find the least-squares solution each problem.
27.43

$$\begin{pmatrix} -2 & -1 \\ 1 & 1 \\ 3 & -1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 0 \\ 14 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 & 3 \\ -1 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 21 \\ 9 \end{pmatrix}.$$

- **<u>ans:</u>**

$$A^T Ax = A^T b$$

$$A^T A = \begin{pmatrix} 14 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 28 \\ -21 \end{pmatrix}$$

$$x = (A^T A)^{-1} A^T b = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$$

$$AA^T y = b \Rightarrow y = \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$$

$$x = A^T y = \begin{pmatrix} -6 \\ 4.5 \\ 1.5 \end{pmatrix}$$

4. (4.1:2a) Find the least-squares solution for the over-
27.44 determined system. Find the root-mean-square-error.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

- **<u>ans:</u>**

$$A^T Ax = A^T b$$

$$A^T A = \begin{pmatrix} 3 & 3 & 2 \\ 3 & 6*3 & \\ 2 & 3 & 3 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 9 \\ 10 \\ 9 \end{pmatrix}$$

$$x = (A^T A)^{-1} A^T b = \begin{pmatrix} 2 \\ -1/3 \\ 2 \end{pmatrix}$$

$$r = b - Ax = \begin{pmatrix} 1/3 \\ 1/3 \\ -1/3 \\ 0 \end{pmatrix},$$

$$\|b - Ax\|_2 = \sqrt{(1/3)^2 + ...} = 0.5774$$

$$RMSE = \|b - Ax\|_2 / \sqrt{4} = 0.2887$$

5. (4.1:12) Approximate the function $z = f(x, y)$ by a
27.50 plane.

x	0	0	1	1	1
y	0	1	0	1	2
$f(x,y)$	3	2	3	5	6

- **<u>ans:</u>** $z = a + bx + cy$. Plug in all data points:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 5 \\ 6 \end{pmatrix}$$

$$A^T A\mathbf{x}x = A^T \mathbf{b}$$

$$\begin{pmatrix} 5 & 3 & 4 \\ 3 & 3 & 3 \\ 4 & 3 & 6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 19 \\ 14 \\ 19 \end{pmatrix}$$

$$\mathbf{x} = \frac{1}{3} \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix}$$

$z = 2 + \frac{5}{3}x + y$.

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1. 5.1: 3ab, 4ab, 6, a1-a2.

30.99
1. (5.1:3) Use the 2-point forward-difference to approxi-
30.84 mate $f'(\pi/3)$ and find the error and the error bound, where

$$f(x) = \sin x, \quad h = 0.1,\ h = 0.01.$$

- **<u>ans:</u>**

Exact

$$f'(x_0) = \cos x_0 = \frac{1}{2}$$

$$f_0' = \frac{f_1 - f_0}{h}$$

$$\begin{aligned} h = 0.1;\ f_0'1 &= \frac{f(\pi/3 + h) - f(\pi/3}{h} = 0.4559 \\ e =& f'(x_0) - f_0' = 0.0456 \\ f''(x) &= -\sin x \\ |e| &\le \frac{h}{2}|f'(c)| \le \frac{h}{2}\sin(\frac{\pi}{3} + h) \\ &= 0.0433 \\ h = 0.01;\ f_0' &= \frac{f(x_0 + h) - f(x_0}{h} = 0.4975 \\ e =& f'(x_0) - f_0' = 0.0043 \\ |e| &\le \frac{h}{2}|f'(c)| \le \frac{h}{2}\sin(\frac{\pi}{3} + h) \\ &= 0.0044 \end{aligned}$$

2. (5.1:4) Use the 3-point central-difference to approximate
30.85 $f'(\pi/3)$ and find the error and the error bound, where

$$f(x) = \sin x, \quad h = 0.1,\ h = 0.01.$$

- **<u>ans:</u>**

Exact

$$f'(x_0) = \cos x_0 = \frac{1}{2}$$

$$f_0' = \frac{f_1 - f_{-1}}{2h}$$

$$\begin{aligned} h = 0.1;\ f_0' &= \frac{f(\pi/3 + 0.1) - f(\pi/3 - 0.1}{2(0.1)} = 0.499167 \\ e =& f'(x_0) - f_0' = 0.00083291 \\ f'''(x) &= -\cos(x) \\ |e| &\le \frac{h^2}{6}|f'''(c)| = \frac{h^2}{6}\cos(\frac{\pi}{3} - h) \\ &= 0.0009732 \\ h = 0.01;\ f_0' &= \frac{f_1 - f_{-1}}{2h} = 0.499991666 \\ e =& f'(x_0) - f_0' = 0.00000833 \\ |e| &\le \frac{h^2}{6}|f'''(c)| = \frac{h^2}{6}\cos(\frac{\pi}{3} - h) \\ &= 0.000008477 \end{aligned}$$

3. (5.1:6) Use the 3-point central-difference to approximate
30.86 $f''(0)$ and find the error, where

$$f(x) = \cos x, \quad h = 0.1,\ h = 0.01.$$

- **<u>ans:</u>**

Exact

$$f''(x_0) = -\cos x_0 = -1$$

$$f_0' = \frac{f_1 - 2f_0 + f_{-1}}{h^2}$$

$$\begin{aligned} h = 0.1;\ f_0' &= \frac{f_1 - 2f_0 + f_{-1}}{h^2} = -0.99916 \\ e =& f'(x_0) - f_0' = -0.00083305 \\ f''''(x) &= \cos(x) \\ |e| &\le \frac{h^2}{12}|f''''(c)| = \frac{h^2}{12}\cos(0) \\ &= 0.00083333 \\ h = 0.01;\ f_0' &= \frac{f_1 - 2f_0 + f_{-1}}{h^2} = -0.999991666 \\ e =& f'(x_0) - f_0' = -0.00000833330 \\ |e| &\le \frac{h^2}{12}|f''''(c)| = \frac{h^2}{12}\cos(0) \\ &= 0.000008333333 \\ h = 0.001;\ f_0' &= \frac{f_1 - f_{-1}}{2h} = -0.99999991665 \\ e =& f'(x_0) - f_0' = -0.0000000833 \\ |e| &\le \frac{h^2}{12}|f''''(c)| = \frac{h^2}{12}\cos(0) \\ &= 0.0000000833 \end{aligned}$$

4. (5.1:a1) Let $f(x) = \cos x,\ x_0 = 0.8.$ Approximate
30.621 $f'(x_0)$.

   (a) Use the central difference with $h = 0.2$. Find the error and the error bound.

   (b) Use the central difference with $h = 0.1$. Find the error and the error bound.

   (c) Use the Richardson extrapolation with the data above. Find the error.

- **<u>ans:</u>**

(a) $h = 0.2$:

$$\begin{aligned} f_0' &= \frac{f_1 - f_{-1}}{2h} \\ f_0' &= \frac{f(x_0 + h) - f(x_0 - h)}{2h} \\ &= \frac{\cos(1) - \cos(0.6)}{0.4} = -0.7126 \\ f'(x_0) &= -\sin(0.8) = -0.7174 \\ err &= f'(x_0) - f_0' \qquad = -0.0048 \end{aligned}$$

Error bound:

$$f''' = \sin x$$

$$\begin{aligned} |f'(x_0) - f_0'| &\leq \frac{h^2}{6} \max x_{-1} \leq z \leq x_1 |f'''(z)| \\ &\leq \frac{0.2^2}{6} (\sin(0.8 + 0.2)) = 0.0056 \end{aligned}$$

Yes! It does bound the error above.

(b) $h = 0.1$:

$$\begin{aligned} f_0' &= \frac{f_1 - f_{-1}}{2h} \\ f_0' &= \frac{f(x_0 + h) - f(x_0 - h)}{2h} \\ &= \frac{\cos(0.9) - \cos(0.7)}{0.2} = -0.7162 \\ f'(x_0) &= -\sin(0.8) = -0.7174 \\ err &= f'(x_0) - f_0' \qquad = -0.0012 \end{aligned}$$

The new error must be smaller!

Error bound:

$$f''' = \sin x$$

$$\begin{aligned} |f'(x_0) - f_0'| &\leq \frac{h^2}{6} \max x_{-1} \leq z \leq x_1 |f'''(z)| \\ &\leq \frac{0.1^2}{6} (\sin(0.8 + 0.1)) = 0.0013 \end{aligned}$$

Yes! It does bound the error above.

(c) We have a second order method:

$$\begin{aligned} f_0' &= \frac{2^2 f_0'(h = 0.1) - f_0'(h = 0.2)}{2^2 - 1} \\ &= -0.717353 \\ err &= f''(x_0) - f_0' = -0.0000024 \end{aligned}$$

Yes. The new error is much much smaller.

5. (5.1:a2) Let $f(x) = \cos x$, $x_0 = 0.8$. Approximate
30.622 $f''(x_0)$.

(a) Use the central difference with $h = 0.2$. Find the error and the error bound.

(b) Use the central difference with $h = 0.1$. Find the error and the error bound.

(c) Use the Richardson extrapolation with the data above. Find the error.

- **ans:** Exact

$$f''(x_0) = -\cos x_0 = -0.6967$$

(a) $h = 0.2$:

$$f_0'' = \frac{f_1 - 2f_0 + f_{-1}}{h^2} = -0.6944$$

error

$$e = f'(x_0) - f_0' = -0.0023$$

Error bound:

$$\begin{aligned} |e| &\leq \frac{\max_{x_0 - h \leq z \leq x_0 + h} |f^{(4)}(z)|}{12} h^2 \\ f^{(4)}(z) &= \cos z \\ |e| &\leq \frac{|\cos(0.6)|}{12} 0.2^2 \\ &= 0.0027 \end{aligned}$$

Yes. The bound is correct, "bigger" than the actual error!

(b) $h = 0.1$:

$$f_0'' = \frac{f_1 - 2f_0 + f_{-1}}{h^2} = -0.6961$$

The error is

$$e = f''(x_0) - f_0'' = -0.0006$$

Error bound:

$$\begin{aligned} |e| &\leq \frac{\max_{x_0 - h \leq z \leq x_0 + h} |f^{(4)}(z)|}{12} h^2 \\ f^{(4)}(z) &= \cos z \\ |e| &\leq \frac{|f^{(4)}(0.7)|}{12} h^2 \\ &= \frac{|0.7648|}{12} h^2 = 0.0006 \end{aligned}$$

Yes. The bound is correct, "bigger" than the actual error!

(c) Extrapolation (2nd order method):

$$\begin{aligned} f_0'' &= \frac{4f_0''(h = 0.1) - f_0''(h = 0.2)}{4 - 1} \\ &= -0.6967 \end{aligned}$$

Checking error: new error is much smaller!!!

$$err = -7.7343 \times 10^{-07}$$
